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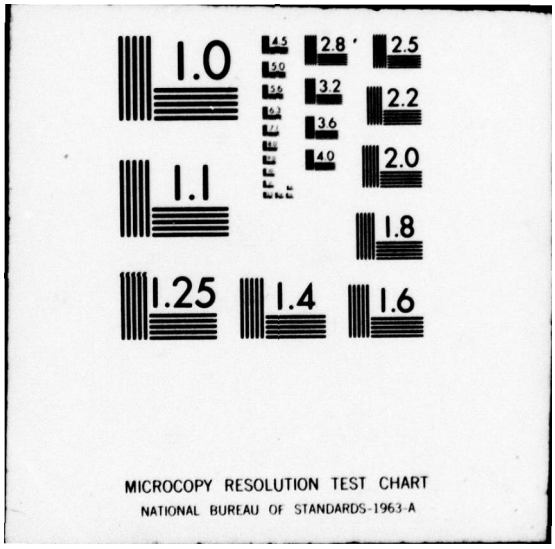
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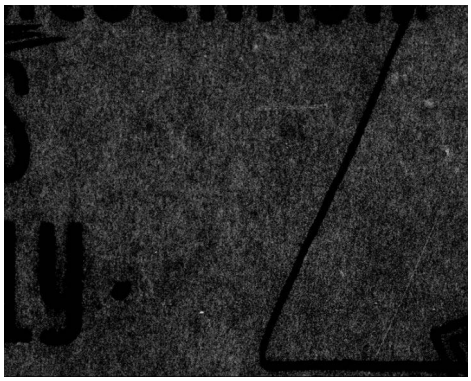

NAVAL RESEARCH LOGISTICS QUARTERLY

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The Naval Research Logistics Quarterly is published by the Office of Naval Research in the months of March, June, September, and December and can be purchased from the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 20402. Subscription Price: [illegible] Cost of individual issues may be obtained from the Superintendent of Documents.

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CONVEX/STOCHASTIC PROGRAMMING AND MULTILOCATION INVENTORY PROBLEMS

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ABSTRACT

This paper examines a convex programming problem that arises in several contexts. In particular, the formulation was motivated by a generalization of the stochastic multilocation problem of inventory theory. The formulation also subsumes some "active" models of stochastic programming. A qualititative analysis of the problem is presented and it is shown that optimal policies have a certain geometric form. Properties of the optimal policy and of the optimal value function are described.

The problem studied in this paper is a special case of the minimization of a convex function under linear constraints. The formulation is motivated by the multilocation distribution problem of inventory theory. Much of the literature in the multilocation area assumes a special structure in the distribution system—typically a multiechelon structure is assumed and restrictions are placed on the available activities. The model studies here is quite general and will allow shipment between all locations, disposal activities, capacity constraints, and multiple products. In effect, no special assumptions need be made about the nature of the constraint-coefficient matrix.

The paper on stock redistribution by Allen [2] is perhaps the earliest examination of a general multilocation problem. A more general version of the distribution problem was posed by Gross [10], and a complete specification of the optimal policy was obtained for a two-location example. Krishnan and Rao [16] have tackled a one-period problem similar to that of Gross except that transshipment decisions are deferred till after demand is realized. Elmaghraby has studied a stochastic transportation problem [8] in which stocks of a product at source locations are to be allocated to a separate set of demand locations. The same problem was studied by Williams [26], who later extended his study to a more general stochastic programming formulation. The stochastic transportation problem has also been studied by Balachandran [3]. A problem similar to that of Gross has been studied by Das [7] with the ordering and transshipment decisions made separately; he derives the optimal policy for the two-location case. Finally, Karmarkar and Patel [14, 15] and Karmarkar [11, 12] have studied the general problem posed by Gross using the methods described in this paper. The results obtained by Gross are recovered and several other examples of two-location problems are solved. They also extend the results to larger problems and demonstrate a computational (column generation) method suggested by the approach on a 5-location problem first posed by Skeith [22].

The problem studied in this paper is more general than the models mentioned above. Other published multilocation inventory models are more specialized than those above and will not be discussed here. They are described in extensive surveys of the literature in multilocation inventory theory by Clark [4], Aggarwal [1], and Karmarkar [11].

In another sense, the problem can be thought of as a generalization of the stochastic programming model studied by Williams [27, 28]. Qualitatively, the formulation also subsumes the two-stage stochastic programming models of Dantzig [6], Wets [24, 25], Walkup and Wets [23], and others, although the special features of two-stage problems are not explored here. The next section presents the problem to be studied, and gives examples to show its relation to stochastic programming. The rest of this paper consists of the qualitative study of this convex/stochastic programming problem using the tools of convex analysis. The notation, terminology, and general approach followed are those of Rockafellar [19] and are briefly introduced in the appendix.

CONVEX/STOCHASTIC PROGRAMS

Consider the convex programming problem

$$(CP) \qquad v(x) = \inf_{z,y}\{cz + L(y)\}$$

$$\text{subject to (s.t.)} \quad Az = y - x,$$

$$z \geqslant 0.$$

Here, x and y are n-vectors, c and z are m-vectors and A is an $(n \times m)$ real matrix. The function L from R^n to the extended reals is a closed, proper convex function defined over all of R^n although dom L may be some proper subset of R^n.

This formulation can be interpreted in terms of the one-period distribution problem in a natural way. Let x be a vector of initial stocks held at n different locations. A stocking decision is to be made which consists of specifying a "target" stock vector y and the set of activities z by which this level is to be reached. The cost of achieving the target level is given by cz. On observation of the demand in the period, a cost of $L(y)$ is incurred. An example of this type of problem is given in Karmarkar and Patel [15]. Suppose that in addition there are constraints on the activities z given by $A_2 z \leqslant b$ and that y is to be restricted to some set Y. Then the problem can be converted to the given form by adding slacks s to the new constraints and by defining $z^1 = (z,s)$, $A^1 = \begin{pmatrix} A & 0 \\ A_2 & I \end{pmatrix}$, $y^1 = (y,\hat{y})$, $x^1 = (x,0)$, $L(y^1) = L(y) + \delta(y|Y) + \delta(\hat{y}|\hat{y} = b)$. Other examples of this type of problem are:

EXAMPLE 1: Let $L(y) = \sum_i l_i(y_i, \bar{y}_i)$, where $l_i(y_i, \bar{y}_i) = \alpha_i(y_i - \bar{y}_i)^+ + \beta_i(y_i - \bar{y}_i)^-$. This formulation is called *goal programming.*

EXAMPLE 2: Let $L(y) = \sum_i E_{\xi_i} l_i(y_i, \xi_i)$, where $l_i(\cdot,\cdot)$ is as defined above and ξ is a random vector. This is termed stochastic programming under *simple linear recourse* and is the problem studied by Williams in [26].

EXAMPLE 3: Let $L(y) = \sum_i E_{\xi_i} g_i(y_i, \xi_i)$; ξ is a random vector and each $g_i(\cdot, \xi_i)$ is a convex function of its first argument for any fixed ξ_i. This case will be called stochastic programming with *simple convex recourse.*

EXAMPLE 4: Let $L(y) = C(y-d)$, where d is a given n-vector and $C(\cdot)$ is defined by $C(x) = \min_{z'} c'z'$, subject to $A'z' = x$; $z' \geqslant 0$. This is an example of a multistage or *dynamic* linear program.

EXAMPLE 5: Let $L(y) = E_\xi C(y - \xi)$, where ξ is a random vector and $C(\cdot)$ is as defined above. This is called the *complete recourse* stochastic programming problem.

EXAMPLE 6: Let $L(y) = L'(y) + \alpha E_\xi v(y - \xi)$; where $L'(y)$ is a closed, proper convex function, $O \leqslant \alpha \leqslant 1$ is a discount factor, and $v(\cdot)$ is as defined in (CP) above. This problem will be called a *multi-stage convex/stochastic program* and is discussed by Karmarkar [11, 13] in greater detail.

The following sections contain a qualitative analysis of the problem (CP) aimed at investigating the structure of the problem, characteristics of the solution set and the nature of the function $v(x)$.

DUALITY & OPTIMALITY CONDITIONS

It can be seen immediately that the problem (CP) can be rewritten in terms of the vector y as an "unconstrained" minimization problem

$$(UCP) \qquad v(x) = \inf_y \Big\{ C(y-x) + L(y) \Big\},$$

where $C(y-x)$ is implicitly defined by the linear programming subproblem

$$LP(y-x) \qquad C(y-x) = \inf_z \Big\{ cz:\ Az = y - x,\ z \geqslant 0 \Big\}$$

or equivalently by the dual

$$LD(y-x) \qquad C(y-x) = \sup_\pi \Big\{ \pi(y-x):\ \pi A \leqslant c \Big\}.$$

The representation in (UCP) follows since the choice of vector y in (CP) leaves z to be chosen exactly as in $LP(y - x)$. This step is termed a *projection* and does not assume any special structure in the problem,

We also define the related convex subproblem (CS) as

$$CS(\pi) \qquad -L^*(-\pi) = \inf_y \Big\{ \pi y + L(y) \Big\}$$

and denote a vector optimal in this problem by $y^0(\pi)$. We note that y^0 solves the problem *iff* $\pi y + L(y)$ is subdifferentiable at y^0 and if 0 is in the subdifferential to $\pi y + L(y)$ at y^0. L^* is by definition conjugate to L and is therefore convex.

The problem as posed, without any assumptions, may not possess a solution, it may be unbounded, or if bounded the infimum may not be attained. To examine the structure of the problem in greater detail, consider the Lagrangian:

$$f(\pi,x) = \inf_{y,z\geqslant 0} \{cz + L(y) - \pi(Az - y + x)\}$$
$$= \inf_{z\geqslant 0}\{(c - \pi A)\, z\} + \inf_{y} \{L(y) + \pi(y-x)\}$$

The first term on the right-hand side is zero for $\pi A \leqslant c$ and is $-\infty$ elsewhere. The second term can be written as $-L^*(-\pi) - \pi x$. Therefore

$$f(\pi,x) = \begin{cases} -L^*(-\pi) - \pi x, & \pi A \leqslant c; \\ -\infty & \text{otherwise.} \end{cases}$$

The dual problem (CD) which involves maximizing a concave function over a convex set can be stated as

$$(CD) \qquad d(x) = \sup_{\pi}\{-\pi x - L^*(-\pi)\}$$
$$\text{s.t. } \pi A \leqslant c.$$

This dual problem can also be formulated via Fenchel Duality which has been shown to be equivalent to the Lagrangian approach (Magnanti, [17]). To see this, first let $C^*(\pi)$ be the indicator function to the set of π satisfying $\pi A \leqslant c$, i.e.,

$$C^*(\pi) = \begin{cases} 0, & \pi A \leqslant c; \\ +\infty & \text{otherwise.} \end{cases}$$

PROPOSITION 1: C^* is the conjugate function of C.

PROOF: We have $C(y) = \sup_{\pi} \pi y$

$$\text{s.t. } \pi A \leqslant c$$

if the dual feasible region is assumed to be nonempty. This can also be written as $C(y) = \sup_{\pi}\{\pi y - C^*(\pi)\}$. Hence C is the conjugate of C^* and, since C and C^* are closed, C^* is the conjugate of C. ||

COROLLARY 1: $C(y-x)$ is conjugate to $C^*(\pi) + \pi x$ for fixed x.

PROOF: $C(y-x) = \sup_{\pi}\{\pi y - C^*(\pi) - \pi x\}$. ||

We can now write (UCP) as

$$v(x) = \inf_{y}\left\{C(y-x) - [-L(y)]\right\}$$

and the Fenchel dual to this problem is (Rockafellar [19])

$$(UCD) \qquad d(x) = \sup_{\pi}\left\{-L^*(-\pi) - C^*(\pi) - \pi(x)\right\}$$

The argument of L^* is $-\pi$ since $-L^*(-\pi)$ is the *concave* conjugate of the concave function $-L(y)$ which is defined exactly as in $CS(\pi)$. Applying Fenchel's Duality Theorem (Rockafellar [19]) to (CD), we have

PROPOSITION 2: For the dual programs (CP) and (CD), $v(x) = d(x)$ if either

(a) ri(dom L) $\cap$ dom $C \neq \emptyset$, or

(b) ri(dom $L^* \cap$ (-dom C^*) $\neq \emptyset$.

If (a) holds, then there is some π^0 such that $d(x) = f(\pi^0, x)$, and if (b) holds then there is some y^0 such that $v(x) = C(y^0 - x) + L(y^0)$. If both conditions hold, then the infimum and supremum are both attained and are necessarily finite and equal.

PROOF: The proof follows from direct application of Theorem 31.1 (page 327) in Rockafellar [19]. ||

The minus sign in condition (b) arises because of the sign of the argument of L^* in the dual problem. The proposition shows when an optimum solution exists to the primal and dual problems.

We will now examine the characterization of optimal solutions to the problem. It will be assumed hereafter that the dual interior condition (b) is satisfied so that the optimal value is bounded from below and is attained in the primal problem for some y. Furthermore, there is no duality gap and $d(x) = v(x)$. We note that (b) does not depend on the vector x that parameterizes the problem. Let us also assume that the primal problem (CP) has a feasible solution. This in conjunction with the bound below ensures that the optimum is finite.

The Kuhn-Tucker optimality conditions for (π^0, y^0, z^0) to be optimal in (CP) or (UCP) are given by

(i) $-\pi^0 \in \partial L(y^0)$

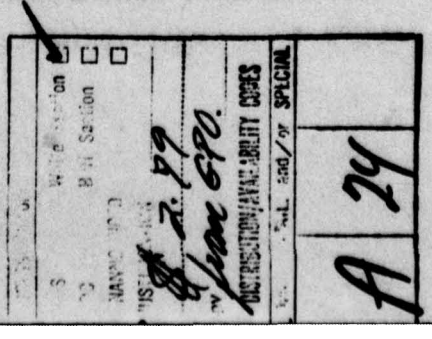
(ii) $\pi^0 A \leqslant c$
$Az^0 = y^0 - x$
$z^0 \geqslant 0$

(iii) $[c - \pi^0 A]\, z^0 = 0$

These conditions are sufficient for optimality in the problem, since it involves minimization of a convex function over a convex set. They are necessary if the primal interior point condition (a) is also met so that the optimum is achieved in the primal and the dual.

It can be seen that (ii) and (iii) are simply the optimality conditions for the dual pair of linear programs $LP(y^0 - x)$ and $LD(y^0 - x)$. Since π^0 is optimal in the dual program, it is a subgradient of $C(\cdot)$ at $(y^0 - x)$. Similarily, the condition (i) applies to a vector y^0 optimal in $CS(\pi^0)$. Thus the optimality conditions can be interpreted as saying that (y^0, π^0) is optimal if $-\pi^0 \in \partial L(y^0)$ and $\pi^0 \in \partial C(y^0 - x)$. We may take the optimal activity vector z^0 to be implicitly given as the solution to the system.

$$Az^0 = y^0 - x,$$
$$[c - \pi^0 A]\, z^0 = 0,$$
$$z^0 \geqslant 0.$$



The solution is unique if the dual problem (LD) is nondegenerate at π^0.

We may also interpret the optimality conditions in terms of the dual problem by examining the dual problem in the "unconstrained" form:

$$(UCD) \qquad d(x) = \sup_{\pi}\left\{-L^*(-\pi) - C^*(\pi) - \pi x\right\}.$$

The optimality conditions for this problem require that there exist a π^0 such that the subdifferential of the function $[-L^*(-\pi) - C^*(\pi) - \pi x]$ at π^0 contains 0. Let $y^0 = y(\pi^0)$, which is the solution to (CS) for $\pi = \pi^0$. Then y^0 is a subgradient of $-L^*$ at $-\pi^0$ when L is closed (Rockafellar [19] p. 218). Given that dom $L^* \cap$ -(dom C^*) is nonempty, we can write that subgradients of the objective function are given by $(y^0 - k - x)$, where k is a subgradient of C^* at π^0. Since C^* is an indicator function, its subgradients are exactly the normal vectors to dom $C^* = \{\pi | \pi A \leqslant c\}$. Hence optimality in the dual requires that $y^0 - k - x = 0$ at some π^0. This says that $y^0 - x = k$, or that $y^0 - x$ must lie in the normal cone to dom C^* at π^0. This implies that π^0 is optimal in $LD(y^0 - x)$; this and y^0 optimal in $CS(\pi^0)$ are equivalent to the Kuhn-Tucker conditions.

EQUIVALENT STOCHASTIC PROGRAMS

In [18], Parikh formulated an equivalent stochastic program to the simple-linear-recourse problem of Williams. The interpretation of the equivalent program was that activities available at the recourse stage could be made available at the initial stage without altering the problem. This section presents a generalization of such equivalent programs for any arbitrary convex L. This construction does not appear to be useful in solving the problem, but it provides some insight into the structure of the problem and, as in Parikh's analysis, may yield qualitative results in particular cases.

The basic idea is simply that adding redundant constraints in the dual does not change the problem. It will be assumed as before that the dual problem (CD) satisfies the interior point condition. This assumption does not depend on the particular value of x, the "initial stock" vector by which the problem is parameterized. The assumption also ensures that $v(x) = d(x)$ and that the optimal in the primal problem (UCP) is attained for some y.

Suppose now that we add to the dual problem (CD) a set of linear constraints $\sigma B \leqslant d$ that are redundant in the sense that $\sigma \in$ dom L^* implies that $\sigma B \leqslant d$. Clearly, the problem with the added constraints is completely equivalent to the original problem. Let us denote the new problem (CD')

$$(CD') \qquad d'(x) = \sup_{\sigma}\{\sigma x - L^*(\sigma)\}$$

$$\text{s.t. } -\sigma A \leqslant c,$$

$$\sigma B \leqslant d.$$

The problem can be written in primal form where $-\sigma = \pi$:

$$(UCP') \qquad v'(x) = \inf_{y}\{C'(y-x) + L(y)\},$$

where

$$LD'(y-x) \qquad C'(y-x) = \max_{\pi} \pi(y-x)$$
$$\text{s.t. } \pi A \leqslant c,$$
$$-\pi B \leqslant d,$$

or by writing $C'(y-x)$ explicitly in primal form,

$$(CP') \qquad v'(x) = \inf_{y,z,u}\{cz + du + L(y)\}$$
$$\text{s.t. } Az - Bu = y-x,$$
$$z,u \geqslant 0.$$

This problem and the original problem (CP) are equivalent in the following sense:

PROPOSITION 3:
(i) $v(x) = v'(x)$
(ii) If (π^0, y^0, z^0, u^0) is optimal in (CP') or (CD') then $(\pi^0, y^0 + Bu^0, z^0)$ is optimal in (CP) or (CD).

PROOF: The first assertion follows from the complete equivalence of the dual problems and from the dual interior point condition (which ensures that no duality gap exists).

It is also clear from the equivalence of the dual problems that the same π^0 will be optimal in both problems. More explicitly, we will show that

(a) π^0 is optimal in $LP(y^0 + Bu^0)$, and

(b) $y^0 + Bu^0$ is optimal in $CSP(\pi^0)$.

The second condition shows that $(y^0 + Bu^0)$ is a subgradient of $-L^*(-\pi^0)$ at $-\pi^0$, and the first condition says that $(y^0 + Bu^0 - x)$ lies in the normal cone to dom C^* at π^0.

To show (a), we use the fact that π^0 is optimal in $LD'(y-x)$. By partial dualization on the added constraints, using the optimal multipliers u^0, we have that π^0 is optimal in

$$\max_{\pi} \pi(y^0 - x) - (-\pi B - d)\, u^0 = \max_{\pi} \pi(y^0 + Bu^0 - x) + du^0$$
$$\text{s.t. } \pi A \leqslant c \qquad\qquad \text{s.t. } \pi A \leqslant c.$$

This proves (a). To prove (b), we first show that Bu^0 is in the normal cone to dom L^* at $\sigma^0 = -\pi^0$. For any $\sigma \in$ dom L, we have $\sigma B \leqslant d$ by hypothesis. Thus, for $u^0 \geqslant 0$ we have $(d - \sigma B)\, u^0 \geqslant 0$. Furthermore, for $\sigma^0 = -\pi^0$ we have $(d - \sigma^0 B)\, u^0 = 0$ by the complementary slackness condition in (LD'). Subtraction gives

$$-\sigma Bu^0 + \sigma^0 Bu^0 \geqslant 0 \text{ or } Bu^0(\sigma^0 - \sigma) \geqslant 0, \quad \sigma \in \text{dom } L^*,$$

which says that Bu^0 is in the normal cone to dom L^* at σ^0. Next we show that if $y^0 \in \partial L^*(\sigma^0)$ and if $\bar{y}$ is in the normal cone to dom L^* at σ^0, then $y^0 + \bar{y} \in \partial L^*(\sigma^0)$. By

hypothesis we have $\bar{y}(\sigma-\sigma^0) \leqslant 0$ for any $\sigma \in \text{dom } L^*$. Furthermore, $L^*(\sigma) \geqslant L^*(\sigma^0) + y^0(\sigma-\sigma^0)$ for any $\sigma \in \text{dom } L^*$. Adding these gives $L^*(\sigma) \geqslant L^*(\sigma^0) + (y^0+\bar{y})\ (\sigma-\sigma^0)$, $\sigma \in \text{dom } L^*$, which says that $(y^0 + \bar{y}) \in \partial L^*(\sigma^0)$. Now, to prove (b), we have that $y^0 \in \partial L^*(\sigma^0)$ by hypothesis and that Bu^0 is in the normal cone to dom L^* at σ^0. Using the result shown above, we have $y^0 + Bu^0 \in \partial L^*(\sigma^0)$, which proves (b). ||

PARAMETRIC ANALYSIS

Two topics of interest in a parametric analysis of the problem are discussed here. One is the form of the optimal policy in the sense of the characteristics of the optimal (π,y) pairs for a given initial state vector x. The second topic is the study of the parametric cost function $v(x)$. A third topic that is not pursued here but is also of great interest is the sensitivity of the cost function $v(x)$ to changes in the cost parameters of the problem. This topic is more problem-specific and is better discussed in the context of some specific numerical algorithm.

The motivation for studying the form of the optimal policy arises from the observation that such qualitative results have typically been useful in inventory theory as well as other fields. Well known examples are the "order-up-to" and "(s,S)" forms of policies that arise in the single-location inventory problem and the "bang-bang" decision rule of control theory. The characteristics of the function $v(x)$ are useful, since the function relates the costs incurred to the initial state x. The properties of $v(x)$ are also important in the study of the multiperiod and infinite horizon problems.

In the following we shall denote by $K^{\prime}_{\pi}$ the "normal cone" to dom C^* at $\pi^{\prime}$. That is to say, $K^{\prime}_{\pi}$ is the set of vectors y such that $(\pi-\pi^{\prime})y \leqslant 0$ for any $\pi \in \text{dom } C^*$. It follows from this that, for any $y \in K^{\prime}_{\pi}$, πy is maximized over dom C^* at $\pi^{\prime}$, since rearranging the inequality above gives $\pi^{\prime}y \geqslant \pi y$, $\forall \pi \in \text{dom } C^*$. Since C^* is the indicator function of a (polyhedral) convex set, it is also known that $K^{\prime}_{\pi}$ is the same as $\partial C^*(\pi^{\prime})$. We note that for any $\pi \in \text{dom } C^*$ we always have $0 \in K_{\pi}$, i.e. $0 \in \partial C^*(\pi)$.

The following proposition shows how optimal (π^0, y^0) pairs can be constructed and identifies the set of initial state vectors x for which the pairs are optimal.

PROPOSITION 4: Suppose that π^0 is dual feasible and L^* is subdifferentiable at $-\pi^0$. Let $y^0 \in \partial L^*(-\pi^0)$. Then (π^0,y^0) is optimal in (CP) – (CD) for all x such that $y^0 - x \in K^0_{\pi}$.

PROOF: (π^0, y^0) satisfies the optimality conditions for (CP) – (CD). ||

Looking at the dual problem (CD) or (UCD) more closely, we can distinguish some special cases:

(i) If $\pi^0 \in \text{int dom } C^*$ and $y^0 \in \partial L^*(-\pi^0)$, then (π^0, y^0) is optimal in (CP) – (CD) only for $x = y^0$. This follows since K^0_{π} in this case consists only of the null vector.

(ii) If π^0 is dual feasible and $y^0 \in \partial L^*(-\pi^0)$, (π^0, y^0) is always optimal for $x = y^0$, since we always have $0 \in K^0_{\pi}$.

(iii) If $-\pi^0 \in \text{int dom } L^*$ it is well known that the set of $y^0 \in \partial L^*(-\pi^0)$ is convex and bounded.

(iv) If $-\pi^0 \notin$ int dom L^*, the set of $y^0 \in \partial L^*(-\pi^0)$ is unbounded. In particular, following the discussion in the proof of Proposition 3, if $y^0 \in \partial L^*(-\pi^0)$ and if $\bar{y}$ is in the normal cone to dom L^* at $-\pi^0$, then $y^0 + \bar{y} \in \partial L^*(-\pi^0)$, and clearly $y^0 + \lambda\bar{y} \in \partial L^*(-\pi^0)$ for $\lambda \geq 0$.

(v) If L^* is not subdifferentiable at $-\pi^0$, π^0 cannot be optimal in the dual problem for any x (by definition) when the interior point condition is assumed to hold.

(vi) Let $S = \{y | y \in \partial L^*(-\pi), -\pi \in \text{dom } L^* \cap (-\text{dom } C^*)\}$. This is the set of all target vectors y generated by applying the subdifferential mapping $\partial L^*(-\pi)$ to dual feasible π. If (π^0, y^0) is optimal in (CP) – (CD) for some x, clearly $y^0 \in S$. Furthermore, from comment (ii) above, if $x = y^0$ the optimal policy is to stay put at y^0. Thus we may think of S as the set of "static" target states.

PROPOSITION 5: If π_1 and π_2 are dual feasible in (CD), $y_1 \in \partial L^*(-\pi_1)$, and $y_2 \in \partial L^*(-\pi_2)$, we cannot have $y_1 - y_2 \in K_\pi^1$; in other words, y_2 cannot lie in the set of points x for which (π_1, y_1) is optimal in (CP) – (CD). Similarly, $y_2 - y_1 \notin K_\pi^2$.

PROOF: We have in general that if, for some convex function f, $\sigma_1 \in \partial f(x_1)$ and $\sigma_2 \in \partial f(x_2)$, then

$$f(x_2) \geq f(x_1) + \sigma_2(x_2 - x_1)$$

and

$$f(x_1) \geq f(x_2) + \sigma_2(x_1 - x_2).$$

Adding and simplifying, we get that

$$(\sigma_1 - \sigma_2)\,(x_2 - x_1) \leq 0.$$

For the case above, we have $(y_1 - y_2)\,(-\pi_2 + \pi_1) \leq 0$ or $(y_1 - y_2)\,(\pi_2 - \pi_1) \geq 0$.

Now if $y_1 - y_2 \in K_\pi^1$ then, by the definition of the normal cone K_π^1, $(y_1 - y_2)\,(\pi - \pi_1) \leq 0$ for any $\pi \in$ dom C^* and hence for any π which is dual feasible. In particular, π_2 is dual feasible, which gives $(y_1 - y_2)\,(\pi_2 - \pi_1) \leq 0$, leading to a contradiction. ||

From an intuitive point of view, the optimal policy can be thought of as a set of optimal or "target" state vectors, such that if the initial state is in the set, then no action is taken. This set of target vectors is generated from the dual feasible region via the subdifferential of L^*. Associated with each target vector y^0 is a set of stocks x for which y^0 is optimal, and this set of stocks is defined by the subdifferential of C^*. From Proposition 5 we know that if a target vector y^0 is in the interior of the "static" set S, then it can only be optimal for $x = y^0$. If π^0 is not in int(dom C^*) then y^0 is optimal for all x such that $y^0 - x \in K_\pi^0$, and the optimal act is to move to state y^0. This is achieved through the set of activities that are optimal in $LP(y^0 - x)$ and the marginal cost of this transition is given by π^0. Examples of this form of policy for distribution problems are given by Karmarkar and Patel [14] and Karmarkar [11].

We next examine the nature of the optimal cost function $v(x)$:

PROPOSITION 6:

(i) $v(\cdot)$ is a closed, proper, convex function.
(ii) dom $v = \{x | x = y - k,\ y \in \text{dom } L,\ k \in -\text{dom } C\}$

(iii) $v(x) \leqslant L(x)$ for all x.
(iv) $v(x) = L(x)$ for $x \in S$
(v) $v^*(\sigma) = L^*(\sigma) + C^*(-\sigma)$
(vi) dom v^* = dom $L^* \cap (-\text{dom } C^*)$

PROOF: From (CD) and (UCD) we see that $v(x) = d(x)$ can be regarded as the conjugate of L^* restricted to (-dom C^*). Assertions (i), (v), and (vi) follow from this observation. Part (iii) follows, since $z = 0$, $y = x$ is always a feasible solution to (CP) and hence $v(x) \leqslant L(x)$. For part (ii), if $x = \bar{y} - k$, $\bar{y} \in$ dom L, $k \in$ dom C then $(\bar{y} - x) \in$ dom C and $v(x) \leqslant C(\bar{y} - x) + L(\bar{y}) < +\infty$ from (UCP). If on the other hand $y - x \notin$ dom C for all $y \in$ dom L then the optimal value in (UCP) must be $+\infty$. Assertion (iv) follows from the comments (ii) and (vi) following Proposition 4.| |

PROPOSITION 7: Let (π^0, y^0) be optimal in (CP) — (CD) for initial state x^0.

(i) $v(x^0) = \pi^0(y^0 - x^0) + L(y^0) = -L^*(-\pi^0) - \pi^0 x^0$,
(ii) $v(x) \geqslant v(x^0) - \pi^0(x - x^0)$ for all x,
(iii) If $y^0 - x \in K_\pi^0$ then $v(x) = v(x^0) - \pi^0(x - x^0)$, and $(\pi^0, y^0$ is optimal in $CP(x)$, $CD(x)$.

PROOF: Part (i) is clear. From (CD) and (UCD) we see that, since L^* and C^* are closed. $-\pi^0 \in \partial v(x^0)$ and hence (ii) follows. For part (iii) we note that (π^0, y^0) satisfy the optimality conditions for (CP) and (CD) for initial state x, and the rest follows using part (i).| |

The following proposition shows that $v(x)$ can be majorized by functions that are piecewise linear over their domains.

PROPOSITION 8:

$$v(x) \leqslant L(\bar{y}) + C'(\bar{y} - x) \leqslant L(\bar{y}) + C(\bar{y} - x), \quad \text{A}\bar{y} \in R^n.$$

PROOF: The first inequality is clear from Proposition 3 (i) and the statement of (UCP'). The second follows from the observation that (LP') is a relaxation of (LP). | |

If $\bar{y} \in S$, the "static set" of states, then equality holds at $x = \bar{y}$, since $v(\bar{y}) = L(\bar{y})$ and $C'(0) = C(0) = 0$. Furthermore, $C'(\cdot)$ and $C(\cdot)$ are positively homogeneous functions of their arguments and are piecewise linear functions, since they represent parametric linear programs.

Finally, let $Y^0(x^0)$ be the set of optimal vectors y for initial state x^0. Since $-\pi^0 \in \partial v(x^0)$, we can write $Y^0(x^0) = -\partial L^*[\partial v(x^0)]$.

AN EXAMPLE

Consider the one-dimensional case with quadratic loss function $L(y) = k(y - y^*)^2$ for some given k, y^*. Suppose that the available activities allow either the increase or disposal of "stock" from the initial state x. The problem may be written as

$$v(x) = \inf\ c^+z^+ + c^-z^- + L(y)$$
$$\text{s.t. } z^+ - z^- = y - x,$$
$$z^+, z^- \geqslant 0.$$

We have

$$C(y - x) = \min\ c^+z^+ + c^-z^-$$
$$\text{s.t. } z^+ - z^- = y - x,$$
$$z^+, z^- \geqslant 0,$$
$$= \max\ \pi(y - x)$$
$$\text{s.t. } -c^- \leqslant \pi \leqslant c^+.$$

Thus we can explicitly exhibit $C(\cdot)$ at the piecewise linear, convex function given by

$$C(x) = \begin{cases} c^+x, & x \geqslant 0; \\ -c^-x, & x < 0. \end{cases}$$

The convex subproblem is

$$-L^*(-\pi) = \inf_y \left\{ \pi y + L(y) \right\}.$$

The optimal solution to this subproblem is $y = y^* - \frac{\pi}{2k}$, and hence $-L^*(-\pi) = \pi y^* - \pi^2/4k$. The conjugate function of C is simply given by

$$C^*(\pi) = \begin{cases} 0 & \text{if } -c^- \leqslant \pi \leqslant c^+, \\ +\infty & \text{otherwise.} \end{cases}$$

Hence the dual problem can be written as

$$d(x) = \sup_\pi (\pi y^* - \pi^2/rk) - \pi x$$
$$\text{s.t. } -c^- \leqslant \pi \leqslant c^+.$$

If we assume that $-c^- < c^+$ (dual interior point condition) then $v(x) = d(x)$, and the infimum in the primal problem is attained. Since the primal problem is always feasible, the infimum is finite.

We can now generate the optimal policy for all initial states x by using the characterizations in Propositions 3 and 5. The set S can be thought of as the set of y such that there is some dual feasible π such that $-\pi \in \partial L(y)$. Once this set is generated, a point in the set is optimal for all x such that $y - x \in K_\pi$. The cones K_π associated with the dual are the positive half line at $\pi = c^+$ and the negative half line at $\pi = -c^-$. Without going into further detail, we can sketch $v(x)$ and the set S in Figure 1. The set S lies between y^+ and y^-, where y^+ is the point at which $\partial L(y^+)$ is $-c^+$, and similarly $c^- \in \partial L(y^-)$. The optimal policy is to do nothing if $x \in S$, to "order" $(y^+ - x)$ if $x < y^+$ and to "dispose of" $(x - y^-)$ if $x > y^-$. This is essentially the familiar "order-up-to" model of inventory theory but with a different loss structure. It may also be thought of as a one-dimensional goal-programming model with quadratic loss.

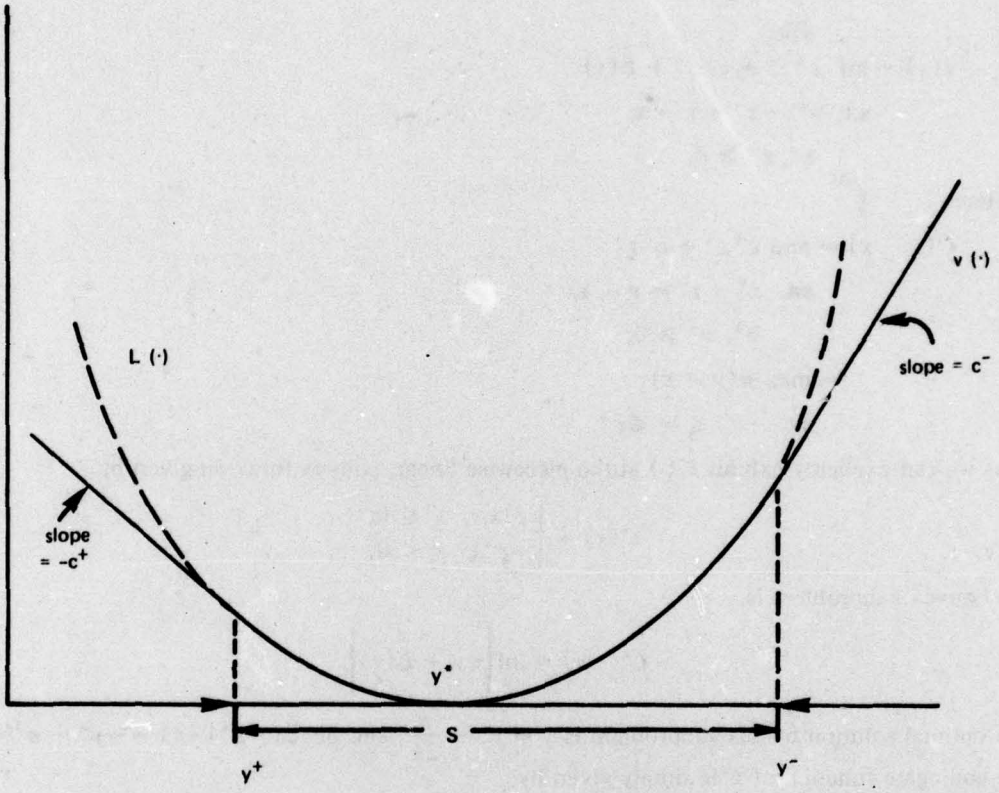


FIGURE 1. Optimal policy and value function
(One variable, quadratic loss example)

A SUBSTITUTION PROPERTY

Williams in [27] makes a comment to the effect that a substitution result similar to that for deterministic Leontieff models also holds for stochastic programs. However, the nature of this property is not clearly specified. A general version of such a property for the model studied here is stated below. It follows as a consequence of previous results on the form of the policy. A particularization of the result for the Leontieff case is then described in a way that makes the connection with the so-called Substitution Theorem clear.

PROPOSITION 9: Suppose that (π^0, y^0) is optimal in (CP) – (CD) for initial state x^0 and that B^0 is the unique optimal basis in $LP(y^0 - x^0)$. Then B^0 is also an optimal basis for any initial state x such that $(y^0 - x) \in K^0_\pi$.

PROOF: This is a restatement of Proposition 7 (iii) with the added requirement that $LD(y^0 - x^0)$ is not degenerate at π^0. This ensures that the same basis B always corresponds to π^0. ||

Note that if $K(B)$ denotes the cone generated by positive linear combinations of the columns of B, then in general $K(B) \subset K^0_\pi$. Furthermore, we can drop the uniqueness requirement in the theorem above if we replace K^0_π by $K(B)$.

The specialization of this result to the Leontieff case is of interest since in uncapacitated multilocation problems, the matrix A generally has a Leontieff structure.

DEFINITION: A matrix A is said to be Leontieff if

(i) Every column of A has exactly one positive entry.
(ii) There is some $z \geqslant 0$ such that $Az > 0$.

Consider the problem $LP(b)$

$$\min cz$$
$$\text{s.t. } Az = b$$
$$z \geqslant 0.$$

The well-known Substitution Theorem (Samuelson [21]) is stated here without proof:

PROPOSITION 10: Suppose A is Leontieff, and that B is an optimal basis in $LP(b^0)$ for some $b^0 > 0$. Then B is optimal in $LP(b)$ for anv $b \geqslant 0$

Let 0^+ denote the nonnegative orthant, i.e., $0^+ = \{b | b \geqslant 0\}$. Furthermore, let π^0 correspond to the basis B, i.e., $\pi^0 = c_B B^{-1}$. We then have the corollaries:

PROPOSITION 11: $0^+ \subset K(B) \subset K_\pi^0$.

PROPOSITION 12: If A is Leontieff there is a unique dual vector π^0 optimal for all $b \geqslant 0$ in $LP(b)$.

PROPOSITION 13: If π is a feasible dual vector in $LP(b)$ then $\pi \leqslant \pi^0$. (Note that dual feasibility does not depend on b.)

There has been considerable interest recently in the properties of constraint sets in Leontieff and "Hidden Leontieff" systems. Some relevant references in this connection are Cottle & Veinott [5], Saigal [20], Yu [29], and Gabbay [9]. The implications for stochastic programs and distribution problems have been discussed by Karmarkar [11].

PROPOSITION 14: Suppose that A is Leontieff and that π^0 is optimal in $LP(b)$ for all $b \geqslant 0$. Furthermore, suppose that L^* is subdifferentiable at $-\pi^0$ and let $y^0 \in \partial L^*(-\pi^0)$. Then y^0 is optimal in CP for all $x \leqslant y^0$.

PROOF: (π^0, y^0) satisfy the optimality conditions for (CP) – (CD). ||

In other words, under suitable assumptions there is a base stock vector y^0 such that whenever stocks are low enough the optimal policy is to raise them to the base stock levels. When the positive orthant 0^+ is strictly contained in K_π^0, the set of starting stocks for which y^0 is optimal may be even larger, as is shown by the following example.

AN INVENTORY EXAMPLE

Consider a two-location system where only location 1 can order exogenously (Z_1) at cost p_1 per unit. Transshipment z_{ij} costs c_{ij} per unit. Starting stocks x_i are available at each location (possibly negative). If after ordering and transshipping y_i is the stock at location i, then $l_i(y_i)$ represents the usual costs of overstocking and understocking after demand is realized (assumed convex). The problem may be stated as follows.

$$\inf p_1 Z_1 + c_{12} z_{12} + c_{21} z_{21} + l_1(y_1) + l_2(y_2)$$
$$\text{s.t. } Z_1 - z_{12} + z_{21} = y_1 - x_1,$$
$$z_{12} - z_{21} = y_2 - x_2,$$
$$Z_1, z_{12}, z_{21}, \geqslant 0.$$

Note that the constraint matrix is Leontieff.

The (LP) subproblem is

$$C(y - x) = \min p_1 Z_1 + c_{12} z_{12} + c_{21} z_{21}$$
$$\text{s.t. } Z_1 - z_{12} + z_{21} = y_1 - x_1,$$
$$z_{12} - z_{21} = y_2 - x_2.$$
$$Z_1, z_{12}, z_{21} \geqslant 0.$$

Its dual is

$$C(y - x) = \max \pi_1(y_1 - x_1) + \pi_2(y_2 - x_2)$$
$$\text{s.t. } \pi_1 \leqslant p_1,$$
$$-\pi_1 + \pi_2 \leqslant c_{12},$$
$$\pi_1 - \pi_2 \leqslant c_{21}.$$

The convex subproblem (CS) decouples into two subproblems:

$$(CS_i) \qquad -l_i^*(-\pi_i) = \inf_{y_i} \pi_i y_i + l_i(y_i).$$

If we assume $c_{12} = c_{21} < p_1$ for the sake of exposition, the feasible region to the (LP) dual can be sketched as in Figure 2. Assume further that the extreme points $(p_1, p_1 + c_{12})$ and $(p_1, p_1 - c_{21})$ are dual feasible. Let y_i represent the optimal solution to $CS_i(\pi_i)$, i.e., $y_i^0 \in \partial l_i^*(-\pi_i)$. Then the stock levels corresponding to these dual feasible points can be computed from the subproblems (CS_i) as in Figure 3. The extreme point $(p_1, p_1 + c_{12})$ in Figure 2 is the Leontieff maximal point and corresponds to the base stock level in Figure 3 which shows the optimal policy for all starting stock conditions. The shape of the static region S depends on the functions $l_i(y_i)$, and the cones are generated as normal cones to the (LP) dual in Figure 2. A detailed account of the computations for such small problems together with several examples can be found in Karmarkar [11] and Karmarkar and Patel [14, 15].

SUMMARY

This paper has described the qualitative analysis of a convex programming problem motivated by the multilocation distribution problem of inventory theory. The formulation also arises in other contexts and in particular subsumes certain "active" stochastic programming models. A decomposition approach is employed, and duality and optimality conditions for the model are discussed. In a parametric analysis of the problem, properties of the optimal value function (perturbation function) are described.

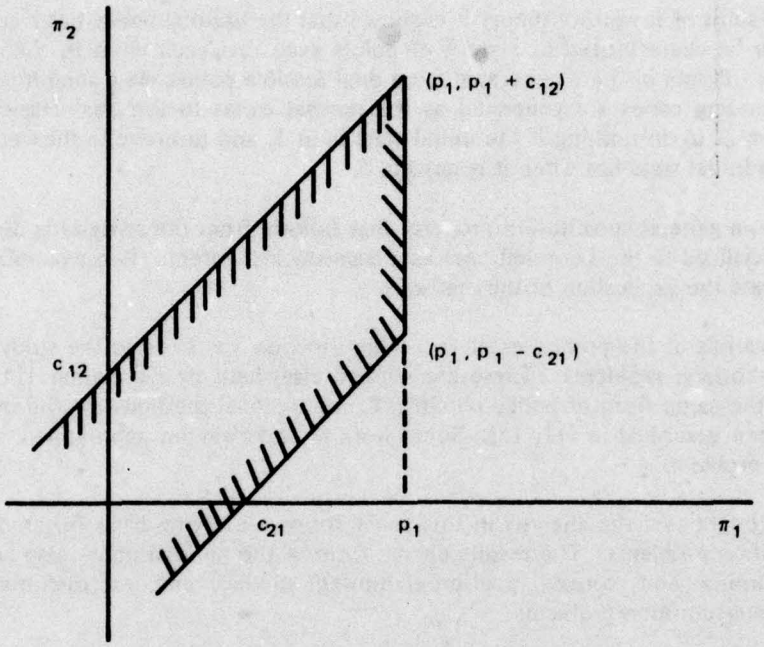


FIGURE 2. LP dual feasible region
(two-location inventory example).

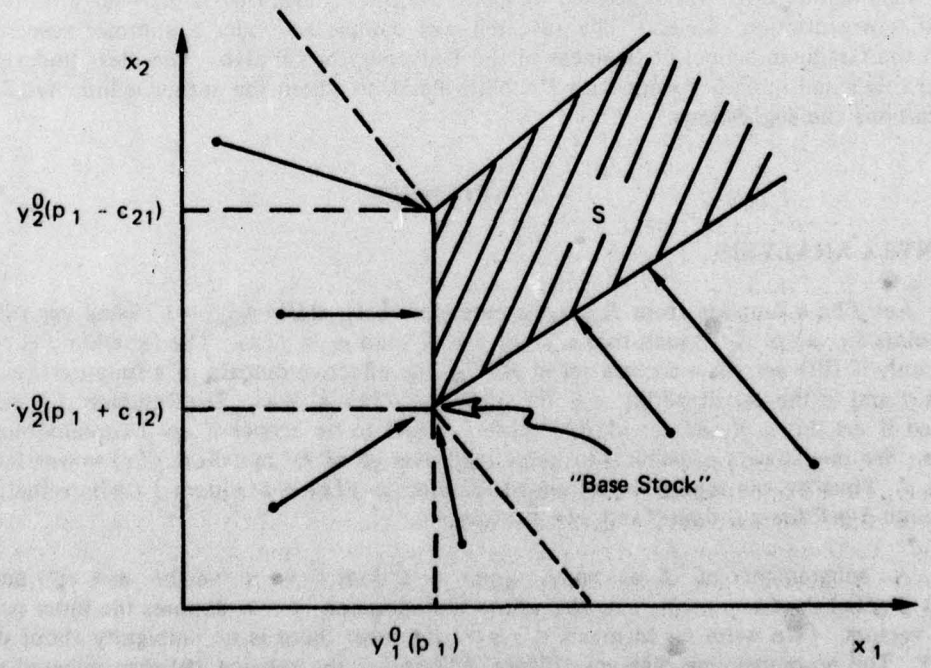


FIGURE 3. Optimal policy for all starting conditions
(two-location inventory example)

In the spirit of inventory theory it is shown that the optimal policy has a certain geometric form—it can be characterized as a set S of points such that each point in S is the vertex of a convex cone. Points of S are generated from dual feasible points via a subdifferential map, and the corresponding cones are generated as the normal cones to the dual feasible points. The optimal policy is to do nothing if the initial state is in S, and to move to the vertex of the cone in which the initial state lies when it is outside S.

Finally, a general substitution property that follows from the analysis is described and the result is specialized to the Leontieff case as a base-stock theorem. Two examples are described to demonstrate the application of the methods.

The analysis of the properties of the value function v is basic to the study of multiperiod and infinite-horizon problems. These are studied elsewhere by Karmarkar [11, 13], and it is shown that the same form of policy obtains. Computational methods for the one-period problem have been described in [11, 15]. Some work is underway on approximate solutions to the multiperiod problem.

It is thought that the analysis in this paper forms a unifying basis for studying multilocation distribution problems. The results on the form of the optimal policy also apply to stochastic programming and convex goal-programming models and to cash-management and investment-consumption problems.

ACKNOWLEDGMENTS

Part of this work was supported by ONR Contract N00014-67-A-0204-0076 on multilevel logistics organization models. The research was completed under a summer research grant from the Graduate School of Business of the University of Chicago. The ideas underlying this paper originated in earlier work with Dr. Nitin Patel, to whom the author is indebted for many discussions and suggestions.

APPENDIX

CONVEX ANALYSIS

Let f be a function from R^n to the extended reals, $RU\{+\infty, -\infty\}$. Then *epi* f is the set of points (x, μ) in R^{n+1} such that $\mu \in R$, $x \in R^n$ and $\mu \geqslant f(x)$. The function f is convex if and only if (iff) *epi* f is a convex set in R^{n+1}. The effective domain of a function f is denoted dom f and is the set of points $x \in R^n$ such that $f(x) < +\infty$. The function f is said to be closed if *epi* f is a closed set in R^{n+1} and f is said to be proper if *epi* f contains no vertical lines. We may always consider f to be defined over all of R^n by taking $f(x) = +\infty$ for all $x \notin$ dom f. Thus we can regard the extended function as $f(x) + \delta(x|\text{dom } f)$ where the indicator function δ is 0 for $x \in$ dom f and $+\infty$ elsewhere.

A subgradient of f at some point $x \in \text{dom } f$ is a vector $\sigma \in R^n$ such that $f(z) \geqslant f(x) + <\sigma, z-x>$, $z \in R^n$, where the operation $<\cdot,\cdot>$ denotes the inner product of two vectors. (We write σy to mean $<\sigma,y>$ whenever there is no ambiguity about the notation). The accompanying diagram (Figure A1) shows the relation between subgradients at a point x and the (nonvertical) supporting hyperplanes to epi f at $[x, f(x)] \in R^{n+1}$. Clearly, if σ is a subgradient of f at x, $(\sigma, -1) \in R^n$ is the normal vector identifying a supporting hyperplane to *epi* f at $[x, f(x)]$, since the latter condition implies that

$$\sigma x - f(x) \geqslant \sigma z - \mu, \text{ for } (z, \mu) \in \textit{epi } f,$$

$$\geqslant \sigma z - f(z),$$

which is the same as the definition of subgradient.

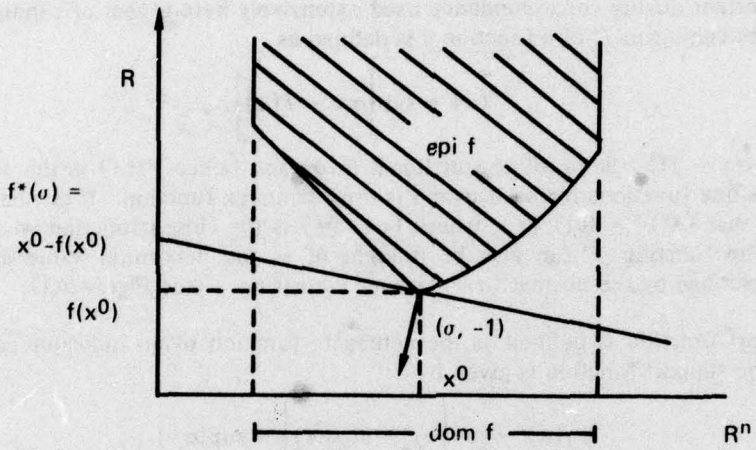


FIGURE A1.

In general there may be more than one subgradient (or supporting hyperplane) at a given point, but if the function is differentiable, then the subgradient is unique and equal to the gradient of the function.

The set of all subgradients of f at x is called the subdifferential of f at x is denoted $\partial f(x)$. The subdifferential of f is the set-valued mapping ∂f: $x \to \partial f(x)$ and the set $\partial f(x)$ is a closed convex set which is always nonempty and bounded for any x in the interior of dom f. A function f is said to be subdifferentiable at a point x if $\partial f(x)$ is nonempty. As an example of a closed, convex function that is not subdifferentiable (Rockafellar [19]), consider f: $R - R$

$$f(x) = \begin{cases} -|(1-x^2)^{1/2}|, & -1 \leqslant x \leqslant +1, \\ +\infty & \text{elsewhere.} \end{cases}$$

Here $f(x)$ is finite (equal to zero) at $x = +1$ and at $x = -1$, but *epi* f does not have a nonvertical supporting hyperplane at these points.

The interior of a set A will be denoted int A and is the set of points $x \in A$ such that there is an open set $N \subset A$ which contains x (there is an open neighborhood of x contained in A). The relative interior of A is denoted ri A and is defined as the interior of the set relative to aff A, which is the smallest affine set containing A, i.e., $x \in$ ri A if $x \in N \subset A$ and N is open relative to aff A. An affine set, it will be recalled, is a set of vectors such that if x and y are in the set, then all points z on the line through them given by $z = \lambda x + (1 - \lambda)y$, $\lambda \in R$, are also in the set. An affine set can always be thought of as a translated subspace, and for every affine set M there is a unique subspace S and $m \in R^n$ such that

$$M = \{y | y = x + m,\ x \in S\}$$

and furthermore

$$S = \{z | z = x - y,\ x, y \in M\}.$$

The closure of a set A will be denoted cl A and the boundary of a set A is given by (cl A)|(int A) = $\{x | x \in \text{cl } A, x \notin \text{int } A\}$. Analogously, the relative boundary of A is (cl A)|(ri A).

An important duality correspondence used extensively here is that of conjugacy of convex functions. The conjugate f^* of a function f is defined as

$$f^*(\sigma) = \sup_x \left\{ \sigma x - f(x) \right\}.$$

For every x, $\sigma x - f(x)$ is an affine functional in σ, and hence $f^*(\sigma)$ is the supremum of a collection of affine functions for each σ and is thus a convex function. It can be shown that f^* is closed and that $(f^*)^* = (\text{cl})\ (f)$, where $(cl)\ (f)$ is the closed function whose epigraph is cl$(epi\ f)$. The function f^* can also be thought of as the maximum value attained by the hyperplane described by the normal $(\sigma, -1)$ over the set $epi\ f$ (see Figure A1).

A support function is defined as the conjugate function of an indicator function. Thus, for a set A, the support function is given by

$$\delta_A^*(\sigma) = \sup_y \left\{ \sigma y - \delta(y|A) \right\} = \sup_{y \in A} \{\sigma y\}.$$

REFERENCES

[1] Aggarwal, S. C., "A Review of Current Inventory Theory and Its Applications," International Journal of Production Research, 12, 443-481 (1974).

[2] Allen, S. G., "Redistribution of Total Stock over Several User Locations," Naval Research Logistics Quarterly, 5, 337-348 (1958).

[3] Balanchandran, V., "The Stochastic Generalized Transportation Problem—An Operator Theoretic Approach," presented at the ORSA/TIMS Joint National Meeting, Boston (1974).

[4] Clark, A. J., "An Informal Survey of Multiechelon Inventory Theory," Naval Research Logistics Quarterly, 19, 621-650 (1972).

[5] Cottle, R. W., and A. F. Veinott, "Polyhedral Sets Having a Least Element," Mathematical Programming 3, 238-249 (1972).

[6] Dantzig, G. B., "Linear Programming Under Uncertainty," Management Science, 1 197-206 (1955).

[7] Das, C., "Supply and Redistribution Rules for Two Location Inventory Systems: One-Period Analysis," Management Science, 21, 765-776 (1975).

[8] Elmaghraby, S. E., "Allocation Under Uncertainty When the Demand Has a Continuous D. F.," Management Science 6, 270-294 (1960).

[9] Gabbay, H., "A Note on Polyhedral Sets Having a Least Element," submitted for publication.

[10] Gross, D., "Centralized Inventory Control in Multilocation Supply Systems," Chap. 3 in *Multistage Inventory Models and Techniques*, H. Scarf, D. Gilford and M. Shelley (eds.) (Stanford University Press, Stanford, California 1963).

[11] Karmarkar, U. S., "Multilocation Distribution Systems," unpublished Ph.D. Thesis, Massachusetts Institute of Technology, Cambridge (1975).

[12] Karmarkar, U. S., "The Multilocation Distribution Problem," presented at the ORSA/TIMS Joint National Meeting, Chicago (1975).

[13] Karmarkar, U. S., "Multiperiod Multilocation Inventory Problems," submitted for publication.

[14] Karmarkar, U. S., and N. Patel, "The N-Location Distribution Problem," presented at the ORSA/TIMS Joint National Meeting, Boston (1974).

[15] Karmarkar, U. S., and N. Patel, "The One-Period, N-Location Distribution Problem," Naval Research Logistics Quarterly, **24**, 559-575 (1977).

[16] Krishnan, K. S., and V. R. K. Rao, "Inventory Control in N Warehouses," Journal of Industrial Engineering, **16**, 212-215 (1965).

[17] Magnanti, T. M., "Fenchel and Lagrange Duality Are Equivalent," Mathematical Programming, **7**, 253-258 (1974).

[18] Parikh, S. C., "Equivalent Stochastic Linear Programs," Journal on Applied Mathematics, **18**, 1-5 (1970).

[19] Rockafellar, R. T., *Convex Analysis* (Princeton University Press, Princeton, New Jersey 1970).

[20] Saigal, R., "On a Generalization of Leontieff Systems," Working Paper, University of California, Berkeley, California (1970).

[21] Samuelson, P. A., "Abstract of a Theorem Concerning Substitutability in Open Leontieff Systems," in *Activity Analysis of Production and Allocation*, T. C. Koopmans (ed.) (Wiley, New York 1951).

[22] Skeith, R. W., "A Multilocation Inventory Model," Journal of Industrial Engineering, **29**, 630-633 (1968).

[23] Walkup, R. W., and R. J. B. Wets, "Stochastic Programs with Recourse," SIAM Journal on Applied Mathematics, **15**, 1299-1314 (1967).

[24] Wets, R. J. B., "Programming Under Uncertainty: The Equivalent Convex Program," SIAM Journal on Applied Mathematics, **14**, 89-105 (1966).

[25] Wets, R. J. B., "Programming Under Uncertainty: The Solution Set," SIAM Journal on Applied Mathematics, **14**, 1143-1151 (1966).

[26] Williams, A. C., "A Stochastic Transportation Problem," Operations Research, **11**, 759-770 (1963).

[27] Williams, A. C., "On Stochastic Linear Programming," SIAM Journal on Applied Mathematics, **13**, 927-940 (1965).

[28] Williams, A. C., "Approximation Formulas for Stochastic Linear Programming," SIAM Journal of Applied Mathematics, **14**, 668-677 (1966).

[29] Yu, P. L., "Cone Convexity, Cone Extreme Points and Non-Dominated Solutions in Decision Problems with Multiobjectives," Journal of Optimization Theory and Applications **14**, 319-377 (1974).

SURVEY OF APPROACHES TO READINESS

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ABSTRACT

About thirty references that feature naval logistics environments are considered. All are unclassified and all appear in the open literature or are available from the Defense Logistics Studies Information Exchange. Three approaches are identified—data analysis, theoretical models, and readiness indexes—and conclusions are presented as to possibilities for answering two questions: (a) Can the unit do the job? (b) How does readiness depend on resources? Four cases are treated in detail to illustrate methodology.

INTRODUCTION AND SUMMARY

There has doubtless always been interest in assessing readiness of military units to carry out particular tasks. Time frames have consisted of immediate instants or extended intervals, methods have ranged from personal judgments to sophisticated calculations, actual assessments have varied from merely "yes" or "no" to indexes and complicated probability statements, and so on, but the primary question has always been: (a) Can the unit do the job? And given the assessment, the question for logistics has been: (b) How does readiness depend on resources? For example, in Part I of the Logistics Research Conference proceedings [21] each of the four senior service representatives referred to these questions in addressing major issues and problems in logistics.

In the present paper we provide a brief survey of several approaches to answers for (a) or (b). We have found that we can, without disadvantage, restrict our attention to unclassified research reports that feature naval logistics environments and appear in the open literature or are available from the Defense Logistics Studies Information Exchange (DLSIE). We start in Section 1 with a concise general review of the contribution and status of about 30 references; we divide these references into three convenient classifications:

(1) Data Analysis
(2) Theoretical Models,
(3) Readiness Indexes.

In Section 2 we present a more detailed discussion of four important cases which are reviewed in Section 1. Our general conclusions are the following.

First, by far the most promising approach to obtaining practical answers to questions (a) and (b) appears to be represented by the methodology study of the Institute of Naval Studies [20] conducted for the Navy Readiness Analysis System; it could be extended by inclusion of further cluster analysis techniques such as those reported by Solomon [28] and pattern recognition procedures. In specific cases, the generation of special data— as typified by the Polaris Military Essentiality System [4] — or the straightforward use of existing authoritative data — as in the Logistics Research Project report [9]— seems worthwhile.

Second, theoretical models, such as represented by Gaver and Mazumdar [11], Kaplan [16], and others noted below, should be continued to be studied in connection with particular problems in which readiness can be involved. They are not to be regarded as immediate sources of operational answers to questions (a) or (b), but their study should produce results that will be helpful in devising practical procedures.

Third, we have found no evidence to indicate that hierarchical models involving the calculation of a readiness index of a system based on the readiness indexes of its components (as in METRI [3] and MARIS [10]) are promising for answering questions (a) or (b). In the first place, requirements for data (especially functional representations) are overwhelming. In the second, it is doubtful that the hierarchies could be used as hoped for, even if data were available.

In terms of what we have been able to survey, it appears that there is no body of data that can be used, together with existing methodology, to answer question (a).

1. GENERAL REVIEW

There are a number of ways in which we can classify different approaches that have been taken to readiness. For example, some efforts fall under the heading *operational readiness*, where attention is mainly focused on operations that the military unit is required to perform. Such efforts often seek results somewhat like *sufficient conditions* in mathematics: given certain evidence, say from training exercises, a result might be a prediction that the unit will be able to accomplish a particular operation. Other efforts fall under the heading *material readiness*, where attention is mainly focused on physical objects. Here results are often sought that are somewhat like *necessary conditions* in mathematics: given certain evidence, say from inspections, a result might be a prediction that the unit definitely cannot accomplish a particular operation. Other terms appear in the literature, for example, combat readiness and industrial readiness, but instead of classifying approaches in this way we use (1), (2), and (3) displayed in the preceding section.

1.1 Data Analysis

All approaches to readiness that we consider involve some kind of analysis of data but here under classification (1) we collect those that depend almost entirely thereon. Here the data evidently have logical connections with what is needed to carry out particular military tasks. The issues that distinguish different approaches are mainly two:

How pertinent are the data?
How defensible are the analyses?

Let us proceed to specific examples.

Early efforts on military worth—synonymously, military essentiality—at the Logistics Research Project illustrate the first approach where special methods produce special applicable data. These efforts were mainly directed at the *ships allowance list problem*, which is the problem of determining the list of quantities of "repair parts" carried on board a ship in direct support of the installed "equipments." Readiness entered explicitly through coverage of question (a) by questionnaires for the determination of specific consequences on the ship's mission following need for the part, when no spare was available. In the most serious case for question (b), the task—say the patrol of a submarine—would have to be terminated. The scheme for submarines given by Denicoff, Fennell, and Solomon [5] is modified by Denicoff et al. [4] for the Polaris weapons system and Denicoff, Haber, and Varley [6] apply the method to naval aviation. Particularly, the Polaris scheme of *military essentiality classes* has had long use by the Navy, both as a source of descriptors— highest worth, high worth, and so on — and for providing "readiness data" inputs for procedures and models connected with inventory problems.

The Navy produces various kinds of status reports (for example, Force Status and Identity Reports, Ready Material Condition data), many of which include readiness grades or "C-ratings" such as

C-1 Fully ready,
C-2 Substantially ready,
C-3 Marginally ready,
C-4 Not ready

Grades are assigned by the individuals responsible for the military tasks, or in some cases for the "equipments," in question. Answers to question (a) are directly given in this fashion and, in cases where "resources" determine C-ratings, responses are also made to question (b) The Logistics Research Project report [9] describes a method of analyzing and using such data for a fleet of destroyers, as follows. For one ship there are C-ratings for eleven subresources covering personnel, supply, equipment, and (average) training. A single C-rating is deduced for each ship using a "weakest link approach" and then average C-ratings are obtained for groups of ships. Measures are also obtained for individual subresources in ways that are responsive to question (b); specifically, the difficulty of improving readiness (by improving particular subresources) is addressed and the major problem areas are identified. In summary, we can say that in [9] the data are by design pertinent and the analyses are intentionally unsophisticated.

The U. S. Navy Board of Inspection and Survey has long been a source of data on the material condition of ships and their readiness. Segel [27] describes origins of a uniform analytical inspection methodology that was in use for a long period and Solomon [28] reports results from cluster analyses on such data. McVoy [23] is a source of considerable information on different approaches to readiness based on physical condition. He also furnishes a substantial list of references. Data on repairs, modifications, and overhauls to ships similarly offer promise of helpful conclusions on the physical condition of ships; for example, Hamilton [13] did an early study of effects of personnel, material supply, availability, obsolescense, and deterioration on readiness.

It is our opinion that by far the most substantial "data analysis" approach to readiness is given in the methodology study of the Institute of Naval Studies [20] for the Navy Readiness Analysis System. It reports on work during the second half of the 1960's, when the Navy was committed under high priority to the development of such a system for, among other things, determining how changes in resources and environments can be expected to affect the performance and capabilities of Navy units, forces, and activities. In the words of its abstract, it

describes a method developed for systematically examining the relationships among personnel, training, equipment, and supply resource variables and destroyer performance measures. Equations for evaluating performance readiness of Atlantic Fleet destroyers at the end of refresher training are presented, and recommendations are made for improving performance measurement and resource data collection.

A broad range of statistical procedures is applied in [20], and in Section 2 of the present paper we provide a short discussion of the methodology. It is important to comment that, in contrast with the approaches reported in Section 1.3, the methodology in [20] does not try to express the readiness of a ship by one index number, but instead provides a vector of readiness score factors which are uncorrelated. Each score factor provides additional information and is thus an important factor of readiness. We believe that anyone who is interested in pursuing readiness analysis should study [20].

1.2 Theoretical Models

In every approach to readiness that we have found, there is a model of some kind, and somewhere there is theory, but here we collect efforts designed to provide models and analytical solutions to specific problems that may be related to the evaluation of readiness. We do not include theoretical models on inventory, maintenance, replacement, reliability, and so on, even though such matters affect important aspects of readiness evaluation. We include only models directly motivated by the problem of assessing readiness.

One of us has surveyed in the Logistics Research Conference proceedings [31] the problem of measuring and making a statistical inference on operational readiness. The papers that are surveyed— Gaver and Mazumdar [11], Mazumdar [22], and Zacks [30]— consider the model of a two-state Markov chain ("up" and "down") and the problems are to estimate the probability of readiness in particular ways. Another example is Tolins [29], who works with a time series of readiness grades for an individual ship as a continuous—time Markov chain having stationary transition probabilities, and then deals with groups of ships.

Several papers on readiness and related areas were prepared at New York University during 1972-75. Barish and Ehrenfeld [1], and Kaplan [14, 16, 18] are concerned with measurement of readiness by a *production function* or *utility function*, similar to those used in economics. These functions constitute readiness data, but they must describe level of performance (output) in terms of available resources (inputs) and, as such, valid ones are difficult to obtain, certainly as compared with any data that we have considered above. Greenberg [12] presents two techniques for measuring readiness. Effects of transportation are studied by Kaplan in [15] while in [17] and [19] he considers replacement problems.

1.3 Readiness Indexes

Under the present classification we collect studies that depend in essential ways on measurements, say on a scale from zero to unity, that can be called *readiness indexes.* In essence, such indexes represent values of functional relationships of the kind described above as production functions and, again, valid examples are difficult to find. The main idea here is to suppose that there is an index for each part of a large (typically hierarchical) complex system and that (by aggregation) the readiness of the system is determined by the value of a single index or by at most a few of them. In other words, the present approach involves indexes that might perform analogously to "gross national product" in economics or "intelligence quotient" in psychology or education.

The METRI Project was sponsored by the U. S. Navy during the early 1960s. Its objective was to develop a system using readiness indexes to measure military essentiality of repair parts for (destroyer) allowance lists. A ship was represented as a hierarchical structure proceeding downward through missions, functional subsystems, and components. The actual hierarchy was to be constructed using four basic structures, series, supplements, alternates, and collaterals, for which rules were given so that readiness indexes for subsystems could be calculated from those for components, indexes for missions from those for subsystems, and so on. In the end, effects of changes in inventory levels for parts were to be transmitted up the hierarchy. We review this project in more detail in Section 2. Dunlap and Associates [7, 8] and Rupp and Kronson [24] provide some details, and Cooper [3] presents afterthoughts.

Project MARIS was a successor to METRI. It addressed the problems of relating the material-support budget and budgetary changes to the operational capability of the Polaris weapons system and of assessing impacts of changes in the logistics support system on the operational capability. It was a very large multiechelon effort involving many data analyses, numerous theoretical models, several simulations, and great complexity. We include it here because an attempt was made to provide a single readiness index to measure the performance of a complex military system. Changes "down below," say in repair parts support, were to be transmitted "to the top" where they were to be read off as changes in the readiness index. Details are to be found in the General Electric Company manual [10]; the methodology is discussed in Section 2.

The MAXCAP model of Schnelker [25, 26] is intended for use in preparing ships' allowance lists. It fits well into approach (3) because it in effect involves a maximization of ships' capability (readiness index) subject to a stipulated budget. It again uses a hierarchical model. But it should be noted that it was an internal Navy effort at the Fleet Material Support Office that was far smaller than the contract efforts in METRI and MARIS.

A motivating factor common to all of these efforts is the need to measure the effects of budgetary changes on the readiness of large-scale complex systems. The studies mentioned above attempted to index readiness as a funtion of factors that are influenced by budgetary constraints. However, the readiness indexes proposed do not attain the desired objective. They are usually very insensitive to changes that occur at the lower echelons and, furthermore, they are generally improper indexes of readiness.

2. METHODOLOGY

In this section we discuss four studies, two from Section 1.1 and two from 1.3.

2.1 Two Examples of Data Analysis

Let us again consider the Logistics Research Project report [9], where every ship is represented as a collection of eleven subresources (propulsion, navigation, communication, weapon systems, personnel, and so on). Each subresource is given a grade by the commanding officer, C-1, C-2, C-3, or C-4, as previously described. The question in [9] is how to analyze the vectors of eleven grades obtained periodically from each ship to obtain a pattern of readiness for the individual ships and for the entire fleet. A methodology for such an analysis is proposed based on conversion of the C-ratings grades to numerical scores by assignment of values 0 to C-1, 1 to C-4, and p_1 and p_2 $(0 < p_1 < p_2 < 1)$ to C-2 and C-3, respectively. (These numerical scores reflect the state of unreadiness rather than the state of readiness.) The state of readiness of the whole ship is expressed as the minimal C-grade of the subresources (the worst readiness rating). The state of unreadiness of the whole fleet is

expressed as an average of the unreadiness of the individual ships. This average does not reflect the extent of unreadiness in the sense of how difficult it is to improve readiness (or, in other words, how many subresources should be improved before readiness is improved). For the purpose of obtaining this additional information, a fleet measure T is constructed in the following manner. A subresource of a given ship is called *visible* if it agrees with the total rating of the ship. Let M_{ij} be the number of ships in the fleet having a visible ith subresource, being equal to C-j ($i = 1,\ldots,11$; $j = 1,\ldots,4$). A total fleet score for the ith subresource is defined then as

$$v_i = M_{i1} \cdot 0 + M_{i2}p_1 + M_{i3}p_2 + m_{i4} \cdot 1$$
$$= p_1 M_{i2} + p_2 M_{i3} + M_{i4}; \quad i = 1,\ldots,11.$$

The measure of difficulty for improving the state of unreadiness is $T = \sum_{i=1}^{11} v_i$.

The method discussed above is an attempt to quantify the qualitative C-ratings of ships and to measure the state of unreadiness of the fleet by proper averages of the indexes obtained. The quantification method depends on arbitrarily assigned p_1 and p_2 values for the categories C-2 and C-3. In addition, the indexes are based on the minimum rating value of the eleven subresources of a ship. This measurement of unreadiness may lose important information concerning the type of subresources that cause low readiness values. Different ships may be classified as having the same readiness level although their readiness problems may be substantially different. There is some doubt as to whether or not the assignment of C-ratings is an effective evaluation method, and there also are concerns for the reliability of the grades provided by the officers concerned. These questions deserve special study.

The methodology for the Navy Readiness Analysis System in the Institute of Naval Studies report [20] gives procedures for expressing the level of readiness of Navy destroyers as certain functions of the Refresher Training Operational Readiness Inspection, briefly ORI, scores. The study involves 82 destroyers and is designed to analyze the relationship between resource variables and performance scores. The ORI scores relate to 29 areas, of which 21 are related to mission functions, as antiair warfare, antisubmarine warfare, surface warfare, command and control communications, mobility, and casualty control. The resource areas considered are personnel, training, equipment, and supply. Thus, the original performance data consist of 82 vectors (one for each ship) of 21 components. Each component (an ORI score) is provided by a team of inspectors. As anticipated, the 21 scores of the various subsections of a ship are correlated and some subsections are highly correlated. By applying *principal component analysis* (see [20] for special details or Cooley and Lohnes [2] as a general reference) the scores of the 21 subsections are reduced to eight linear combinations, with weights given by the eigenvectors corresponding to the largest eigenvalues. Factor analysis is then performed and a rotated three-factor system provides the most interpretable solution. Factor 1, named *control procedures*, involves six performance variables concerning tactical information (resulting from the interpretation of radar data). Factor II, named *casualty control procedures*, involves four performance variables concerned with procedures of preventing damages and effecting repairs. Factor III, named *antisubmarine warfare tactical communications*, is defined by three performance variables which measure a series of activities with the chain of communications. The performance of each ship is then expressed by three values corresponding to the three factor scores. Ships can be clustered into homogeneous groups according to these three factor scores. The dimensionality of the data has been reduced from 21 correlated variables to three uncorrelated factors.

An important question is how the four resources, personnel, training, equipment, and supply, affect the readiness-factor scores. For this purpose multiple-regression analysis is performed for each one of the factor scores on the various variables characterizing the four resource categories. This analysis shows the relative importance of the various resources on performance-readiness factors. It can provide information on possible interactions between different resource categories (personnel and training, equipment and inventory management, and so on). In addition, the regression analysis provides the means for readiness estimation, given the status of the resource variables. For specific details see the Institute of Naval Studies report [20].

2.2 Two Examples Based on Hierarchical Structures

The methodology of the METRI project as reported by Cooper [3] and by Dunlap and Associates [7,8] was to construct a huge hierarchical structure modeling a Navy ship (destroyer) and to compose a readiness index from readiness values of its elementary units by certain rules. Readiness indexes are to be constructed for each component (elementary unit) according to the *capability* of its parts to function properly throughout the mission period. These indexes are to be functions of the *reliability* of the parts (the failure process) and the number of spare *replacement* parts available. Let $R_1, \ldots, R_{n_i}$ denote the readiness indexes of the components in the ith subsystem ($i = 1,\ldots,k$); then the readiness index of the subsystem is a function

$$R_{S_i} = \phi_i(R_1, \ldots, R_{n_i}), \quad i = 1, \ldots, k.$$

The readiness of the whole system is a function $R_T = f(R_{S_1}, \ldots, R_{S_k})$. The problem (apparently insolvable) is to determine suitable functions for the composition of the readiness indexes to serve as an overall index. For this purpose it was assumed that the hierarchical structure of a ship can be uniquely described as a combination of the following four basic structures:

(i) If $R_1, \ldots, R_n$ are the readiness indexes of n components connected in *series*, then the readiness of the structure is

$$R_T = \left(\prod_{i=1}^{n} R_i^{\alpha_i} \right)^A,$$

where α_i and A are empirical coefficients for the specific items.

(ii) *Supplement structure:* If n items independently supplement one another (for example, sonar, surface radar, and air radar for detection of enemies) then

$$R_T = \left(\sum_{i=1}^{n} K_i R_i^{\alpha_i} \right)^A.$$

The parameters K_i, $\alpha_i (i = 1, \ldots, n)$ provide for the relative importance of the items.

(iii) *Alternative structure:* If a main system whose readiness is R_1 has a standby unit (readiness R_2) to replace it in case it fails, then

$$R_T = R_1 + (1 - R_1) K R_2 .$$

The parameter K expresses the relative ability of the standby unit to replace the main one.

(iv) *Collateral structure*: Let R_1 denote the readiness of a unit that is essential to the operation of a ship. Assume that there is a collateral element that affects the system's readiness in the presence of the essential unit. For example, the collateral element might provide for the maintenance of the essential unit. Let R_2 denote the readiness index of the collateral unit. The readiness of this structure is given by

$$R_T = R_1[K + (1 - K)R_2].$$

In summary, it was supposed that by applying the rules for calculating the readiness of the basic structures one can calculate the readiness of a ship.

A sensitivity analysis was proposed to show the rate of change in the overall readiness as a function of changes in the number of spare parts assigned. Such an analysis was designed to answer the question of the effect of changes in the inventory levels on the readiness of a ship. The proposed analysis is, however, vague in publications on METRI. The whole approach appears to have been found to be theoretically invalid and practically intractable.

As mentioned earlier, the main objective of the General Electric Company project MARIS [10] was to relate the system of material support to the operational capability of the Polaris weapons system. The readiness index was the expected proportion of operational missiles in a specified period of time.

The system considered is a three-echelon support system that contains four squadrons of submarines (first echelon) with one squadron assigned to each of four tenders (second echelon). The tenders reorder from stock points which procure material from outside sources. The stock points, the inventory control points, the repair facilities, and industrial sources constitute the third echelon. Superimposed on the above three-echelon structure is a transportation system for moving material among the various system elements. Routine replenishment is provided by four cargo ships, one assigned to each of the tenders, while occasional high-priority transportation is also available.

The basic MARIS procedure aims to relate the budget for replenishment to readiness of submarines via a budget model, a three-echelon simulation model, and a submarine-readiness model.

The budget model simulates the estimated procurement expenditure for parts. The three-echelon simulation model provides a detailed representation of the Navy support system and it simulates actions taken and resulting effects of all possible events. The submarine-readiness model is an analytic model for the evaluation of the readiness of a submarine.

The submarine-readiness model is the essential part of the project. This model determines the readiness of a submarine as a function of the onboard inventory of spare parts. Let us give a simple example to show how the readiness is calculated. The spare parts treated are related to missiles and are replaceable on patrol. Suppose that a certain part has at the beginning of a patrol n units in stock and is installed in m different applications. For the sake of simplicity, it is assumed that the part has exponentially distributed independent life times at the various applications, with intensity parameters $\lambda_1, \ldots, \lambda_m$. If t denotes the first instant of stockout, the probability distribution function of t is the gamma $g(t|\bar{\lambda}, n)$ where $\bar{\lambda} = \sum_{j=1}^{m} \lambda_j$; that is,

$$g(t|\bar{\lambda}, n) = \frac{\bar{\lambda}}{\Gamma(n)} t^{n-1} e^{-\bar{\lambda}t}, 0 \leqslant t \leqslant \infty .$$

Given that stockout occured at time t, then the probability that all m units will still be operating y units of time after stockout is given by

$$\prod_{j=1}^{m} e^{-\lambda_j y} = \exp(-\bar{\lambda} y)$$

The readiness index conditional on the time t of stockout was defined as

$$R_T(t) = \frac{t}{T} + \frac{1}{T}\left[\int_0^{T-t} \exp(-\bar{\lambda} y)\, dy\right], \quad t \leqslant T,$$

$$= 1, \qquad t > T.$$

Notice that the product of T and $R_T(t)$ equals the conditional expected length of life of the system in a patrol, given that a stockout occurred at time t. Finally, the readiness index related to this part with n units in stock is calculated by randomization, as follows:

$$\bar{R}_T(n) = \int_0^{\infty} R_T(t) g(t|\bar{\lambda}, n)\, dt\ .$$

Now $\bar{R}_T(n)$ is a reasonable index of readiness as a function of a particular part. But the question is, how can one combine these indexes? No satisfactory answer appears to have been given.

REFERENCES

[1] Barish, N.N., and S. Ehrenfeld, "Utilities Estimated from Actual Decisions in Readiness Measurement," Technical Report No. 14, Graduate School of Public Administration, New York University, New York, N.Y. (1975) {LD 35213A}.*

[2] Cooley, W.W., and P.R. Lohnes, *Multivariate Data Analysis* (Wiley, New York, 1971).

[3] Cooper, G., "METRI and the Allowance List Problem," Management Science *13*, B293-326 (1967).

[4] Denicoff, M., J. Fennell, S.E. Haber, W.H. Marlow, F.W. Segel, and H. Solomon, "The Polaris Military Essentiality System," Naval Research Logistics Quarterly *11*, 235-257 (1964).

[5] Denicoff, M., J.P. Fennell, and H. Solomon, "Summary of a Method for Determining the Military Worth of Spare Parts," Naval Research Logistics Quarterly *7*, 221-234 (1960).

[6] Denicoff, M., S.E. Haber, and T.C. Varley, "Military Essentiality of Naval Aviation Repair Parts," Management Science *13*, B439-453 (1967).

[7] Dunlap and Associates, "METRI Pilot Program Report, USS Ellison (DD-864)" (Darien, Connecticut, 1964) {LD 08174A}.

[8] Dunlap and Associates "METRI Personnel Readiness Measurement (Darien, Connecticut, 1965) {LD 08175B}.

[9] Frank, S.A., W.B. Gruttke, W.H. Marlow, and S.J. Mathis, "Readiness Measurements via Subresource C-Ratings," Technical Paper Serial T-216, Logistics Research Project, The George Washington University, Washington, D.C. (1968) {LD 19644}.

[10] General Electric Company, "MARIS Technical Manual," (TEMPO, Santa Barbara, California, 1969) {LD 32578 MA}.

[11] Gaver, D.P., and M. Mazumdar "Statistical Estimation in a Problem of System Reliability," Naval Research Logistics Quarterly *14*, 473-488 (1967).

*LD numbers identify documents that can be obtained from the Defense Logistics Studies Information Exchange, U. S. Army Logistics Management Center, Fort Lee, VA 23801.

[12] Greenberg, I., "On the Determination of the "Closeness" to Complete Readiness and of Dynamic Readiness," Technical Report No. 2, Graduate School of Public Administration, New York University, New York, N.Y. (1972) {LD 28693A}

[13] Hamilton, J.E., "Ship Material Readiness," Technical Paper Serial T-145, Logistics Research Project, The George Washington University, Washington, D.C. (1962) {LD 05636}.

[14] Kaplan, S., "A Production Function Approach to the Measurement of Short Term Readiness of Navy Units," Technical Report No. 1, Graduate School of Public Administration, New York University, New York, N.Y. (1972) {LD 28693}.

[15] Kaplan, S., "Readiness and the Optimal Redeployment of Resources," Technical Report No. 4, Graduate School of Public Administration, New York University, New York, N.Y. (1972) {LD 28829}.

[16] Kaplan, S., "Application of Programs with Maximum Objective Functions to Problems of Optimal Resource Allocation," Operations Research *22*, 802-807 (1974).

[17] Kaplan, S. "A Note on a Constrained Replacement Model for Ships Subject to Degradation in Utility," Naval Research Logistics Quarterly *21*, 563-568 (1974) {Cf. LD 32736A}.

[18] Kaplan, S., "An Approach to the Measurement of the Short Term Readiness of Military Systems," Technical Report No. 10, Graduate School of Public Administration, New York University, New York, N.Y. (1975) {LD 34461A}.

[19] Kaplan, S., "Readiness Analysis for Ships Where both Modernization and Replacement are Used to Increase Fleet Readiness," Technical Report No. 13, Graduate School of Public Administration, New York University, New York, N.Y. (1975) {LD 35512A}.

[20] Lockman, R.F., P.H. Stoloff, B.H. Manheimer, J.R. Hardgrave, and W.F. Story, Navy Readiness Analysis System Methodology Study, Vol. 1," Institute of Naval Studies of the Center of Naval Analyses Study No. 27, University of Rochester, Rochester, New York (1970) {Cf. LD 25118 and LD 25118A}.

[21] Marlow, W.H., ed., *Modern Trends in Logistics Research*, (Massachusetts Institute of Technology Press, Cambridge, Massachusetts, 1976).

[22] Mazumdar, M., "Uniformly Minimum Variance Unbiased Estimates of Operational Readiness and Reliability in a Two-State System. Naval Research Logistics Quarterly *16*, 199-206 (1969).

[23] McVoy, J.L., "The Concept of Materiel Readiness as Applied to Naval Vessels," Logistics Management Institute, Washington, D.C. (1970) {LD 26719}.

[24] Rupp, A.E.,Jr., and E.T. Kronson, "Final Report on METRI Data Processing Program," Vitro Laboratories, Silver Spring, Maryland (1964) {LD 08243}.

[25] Schnelker, H.J. "Max-Cap Allowance Model," Alrand Report 48, Application Development Division, Data Systems Support Office, Mechanicsburg, Pennsylvania (1965) {LD 08595}.

[26] Schnelker, H.J., "Max-Cap Allowance Model (Revised)," Alrand Report 48A, Operations Analysis Department, U. S. Navy Fleet Material Support Office, Mechanicsburg, Pennsylvania (1966) {LD 08595A}.

[27] Segel. F.W., "The Methodology for Improved Inspection Procedures by the U. S. Navy Board of Inspection and Survey," Technical Paper Serial T-160, Logistics Research Project, The George Washington University, Washington, D.C. (1964) {LD 07860}.

[28] Solomon, Henry, "A Note on a First Application of Clustering Procedures to Fleet Material Condition Measurements," Naval Research Logistics Quarterly *18*. 415-421 (1971).

[29] Tolins, I.S., "A Continuous Time Markov Process Model of Naval Operational Readiness," Technical Paper Serial T-214, Logistics Research Project, The George Washington University, Washington, D.C. (1968) {LD 19645}.

[30] Zacks, S., "Bayes Adaptive Estimation of Current Operational Readiness Parameters," Technical Memorandum Serial TM-61018, Program in Logistics, The George Washington University, Washington, D.C. (1969) {LD 39954A}.

[31] Zacks, S. "Review of Statistical Problems and Methods in Logistics Research," in *Modern Trends in Logistics Research,* W.H. Marlow, ed. (Massachusetts Institute of Technology Press, Cambridge, Massachusetts, 1976), pp. 227-247.

ADAPTIVE DISPOSAL MODELS*

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ABSTRACT

This paper reconsiders the classical model for selling an asset in which offers come in daily and a decision must then be made as to whether or not to sell. For each day the item remains unsold a continuation (or maintenance cost) c is incurred. The successive offers are assumed to be independent and identically distributed random variables having an unknown distribution F. The model is considered both in the case where once an offer is rejected it may not be recalled at a later time and in the case where such recall of previous offers is allowed.

1. INTRODUCTION

This paper reconsiders the classical model for selling an asset in which offers come in daily and a decision must then be made as to whether or not to sell. For each day the item remains unsold a continuation (or maintenance cost) c is incurred. The successive offers are assumed to be independent and identically distributed random variables having an unknown distribution F. The model is considered both in the case where once an offer is rejected it may not be recalled at a later time and in the case where such recall of previous offers is allowed.

In Section 2 we show how bounds on the optimal policy may be obtained when some partial information about F is available. In particular, we show that if F, the distribution of offers, satisfies the NWUE (new worse than used in expection) property defined as

$$E_F[X - a | X > a] \geqslant E_F[X] \text{ for all } a \geqslant 0,$$

then the optimal policy has a monotonic relationship with the optimal policy in the case where the distribution of offers is exponential with the same mean as F.

*This research was supported in part by the Office of Naval Research under contracts N00014-75-C-0620 with Columbia University, N00014-75-C-0561 with Stanford University, and N00014-68-A-0200-1036 with the University of California, and by the U.S. Army Research Office-Durham under contract DAHC04-74-C-00226 with the University of California.

In Sections 3 and 4 we consider a Bayesian version of this model by supposing that F is known to be one of the distributions $F_1, F_2, \ldots, F_n$ with given initial prior probabilities. In Section 3 we do not allow, and in Section 4 we do allow, the recall of old offers. In both cases we provide bounds on the optimal policy in terms of the optimal policies in the case where it is known which of the F_i is equal to F. This Bayeisan format has previously been considered in [3], which assumed that F was a normal random variable with known variance and imposed a normal prior distribution on the mean of F. As our model imposes no parametric condition on F in the prior distribution, the type of results we obtain are somewhat different than those in [3].

2. INDEPENDENT AND IDENTICALLY DISTRIBUTED OFFERS FROM AN UNKNOWN DISTRIBUTION WITH PARTIAL INFORMATION

If the successive offers were independent and identically distributed random variables having known distribution F, then it is well-known [2] that the policy that maximizes the total expected return, both with and without recall, is to accept an offer x if and only if $x \geqslant x_F$, where x_F is the smallest value such that

$$x_F \geqslant \left[\int_{x_F}^{\infty} x dF(x) - c\right] / [1 - F(x_F)].$$

If F is continuous, this reduces to

$$c = \int_{x_F}^{\infty} (x - x_F)\, dF(x).$$

The optimal expected return is $x_F + c$.

We shall start our by comparing the optimal critical number for two different distributions. To begin we need the following definition:

DEFINITION: For any two probability distributions F and G we say that $F \underset{V}{\leqslant} G$ if

$$\int f(x)\, dF(x) \leqslant \int f(x)\, dG(x)$$

for all increasing convex functions f.

If F and G have the same means, then $F \underset{V}{\leqslant} G$ intuitively means that F has less variability than G.

PROPOSITION 1: If $F \underset{V}{\leqslant} G$ then $x_F \leqslant x_G$.

PROOF: x_G is the smallest value satisfying

$$c \geqslant E_G\,[(X - x_G)^+].$$

Now

$$E_G\,[(X - x_G)^+] \geqslant E_F\,[(X - x_G)^+],$$

since $f(x) = (x - x_G)^+$ is an increasing convex function. Hence,

$$c \geqslant E_F\,[(X - x_G)^+],$$

implying that $x_F \leqslant x_G$.

Proposition 2 is concerned with the return from a nonoptimal policy:

PROPOSITION 2: If $x \leqslant x_F$, then the policy that accepts the first offer that is at least as large as x has a return that is at least $x + c$.

PROOF: To prove the above, consider the expected difference between the optimal policy that uses the critical number x_F and the above policy that uses the critical number x. By conditioning on whether an offer between x and x_F occurs before or after an offer greater than x_F, we see that the expected difference is at most $x_F - x$ in the former case (since the expected return from the optimal policy starting at the time of this offer between x and x_F is equal to $x_V + c - c = x_F$) and it is 0 in the latter case. Hence, the result follows.

DEFINITION: We say that the distribution F, with $F(0-) = 0$, is NWUE if

$$\int_a^\infty \frac{\bar{F}(x)\,dx}{\bar{F}(a)} \geqslant \int_0^\infty \bar{F}(x)\,dx \text{ for all } a \geqslant 0,$$

where $\bar{F}(x) = 1 - F(x)$. (If X is a random variable having distribution F, then the above is equivalent to $E[X - a | X > a] \geqslant E[X]$.)

PROPOSITION 3: If F is NWUE with mean μ, then

$$E(\mu) \underset{V}{\leqslant} F,$$

where $E(\mu)$ is an exponential distribution with mean μ.

PROOF: It is easy to show that $F \underset{V}{\geqslant} G$ is equivalent to

$$\int_a^\infty \bar{F}(x)\,dx \geqslant \int_a^\infty \bar{G}(x)\,dx \text{ for all } a.$$

Thus, we have to show that,

$$\int_a^\infty \bar{F}(x)\,dx \geqslant \mu e^{-a/\mu}$$

whenever F is NWUE. By the definition of NWUE we have

$$\int_t^\infty \frac{\bar{F}(x)}{\mu}\,dx \geqslant \bar{F}(t)$$

or, equivalently,

$$\bar{F}(t) / \int_t^\infty \bar{F}(x)\,dx \leqslant 1/\mu.$$

Integrating both sides of the above from 0 to a completes the proof.

We are now ready for the main theorem of this section.

THEOREM 1: If the unknown distribution F is known to be NWUE and to have mean μ, then

$$x_F \geqslant \bar{x}$$

and the policy which accepts the first offer of at least $\bar{x}$ has return of at least $\bar{x} + c$, where

$$\bar{x} = -\mu \log (c/\mu).$$

PROOF: The result follows immediately from Propositions 1, 2, and 3, since $\bar{x} = x_E$ when E is an exponential distribution with mean μ.

REMARK: One instance in which the distribution of offers would be NWUE is the case in which there are many classes of potential customers and offers from each class follow an exponential distribution. Thus the distribution of offers would be a mixture of exponential distributions and the degenerate distribution at 0 (indicating no offer), and it would thus be NWUE (since a mixture of NWUE random variables is also NWUE) [1].

3. BAYESIAN MODEL WITHOUT RECALL OF PAST OFFERS

In this section we suppose that if an offer is rejected then it can never be accepted in the future. In addition, we suppose that, although the distribution F is not known with certainty, we do know that it is one of the distributions $F_1, F_2, \ldots, F_n$, with given prior probabilities. We say that the state of the system is $(x, \mathbf{P})$ when x is the present offer under consideration and $\mathbf{P} = (P_1, \ldots, P_n)$ is the posterior probability vector, given all the information that we have accumulated up to that point (including the present offer x), as to which of the F_i is the actual distribution.

Also we define $V(x, \mathbf{P})$ to be equal to the expected return from this day onward, given that the state today is $(x, \mathbf{P})$ and we employ an optimal policy. (If we assume as we do, that each of the F_i has a finite variance and $c > 0$ then it can be shown as in [2] that an optimal policy exists.)

The optimality equation thus takes the following form:

$$V(x, \mathbf{P}) = \max \{x, V(\mathbf{P}) - c\},$$

where $V(\mathbf{P})$, which represents the best you can do when the distribution is chosen by the prior probability vector $\mathbf{P} = (P_1, \ldots, P_n)$, satisfies

$$V(\mathbf{P}) = \Sigma P_j \int V(y, T_y \mathbf{P})\, dF_j(y),$$

where

$$T_y \mathbf{P} = [(T_y \mathbf{P})_1, \ldots, (T_y \mathbf{P})_n]$$

and

$$(T_y \mathbf{P})_j = \text{Prob}\{F_j | \mathbf{P}, y\} = \frac{P_j\, dF_j(y)}{\Sigma P_i\, dF_i(y)}.$$

Furthermore, the optimal policy accepts the offer in state $(x, \mathbf{P})$ if and only if

$$x \geq V(\mathbf{P}) - c.$$

PROPOSITION 4: $V(\mathbf{P})$ is a convex function of $\mathbf{P}$.

PROOF: Recall that $V(\mathbf{P})$ represents the best we can do when the distribution is chosen according to $\mathbf{P}$. Now suppose $\mathbf{P} = \lambda \mathbf{P}^1 + (1 - \lambda)\mathbf{P}^2$, for some $0 < \lambda < 1$, and suppose that the distribution to be used is to be chosen according to the following two-stage experiment. First we flip a coin having probability λ of coming up heads. If the coin comes up heads, then we choose the distribution according to the prior probability $\mathbf{P}^1$, and if it comes up tails then we use $\mathbf{P}^2$. Now if we are not told the outcome of the coin flip then the problem is exactly the same as if the distribution was chosen according to $\mathbf{P}$ and thus the best we can do is $V(\mathbf{P})$. On the other hand, if we are to be told about the outcome of the coin flip, then by conditioning on the outcome we see that our expected return if we play optimally is $\lambda V(\mathbf{P}^1) + (1 - \lambda) V(\mathbf{P}^2)$. Hence, as additional information can not lower our expected return, we see that

$$V(\mathbf{P}) \leqslant \lambda\, V(\mathbf{P}^1) + (1-\lambda)\, V(\mathbf{P}^2),$$

and the result is proven.

Let $x_i = x_{F_i}$, $i = 1, \ldots, n$.

COROLLARY 1: $V(\mathbf{P}) \leqslant \Sigma P_i\, x_i + c$.

PROOF: This follows directly from Proposition 4, since $V(0,0,0,1,0,\ldots,0) = x_i + c$ (where the 1 is in the ith place).

PROPOSITION 5: If the present state is $(x, \mathbf{P})$, then

(i) if $x \geqslant \Sigma P_i\, x_i$, it is optimal to accept x,

(ii) if

$$x < \Sigma\, P_i \left[\frac{\int_x^\infty y\, dF_i\,(y) - c}{1 - F_i(x)}\right],$$

it is optimal to reject the offer x,

(iii) if

$$x < \Sigma\, P_i \int_{-\infty}^{\infty} y DF_i\,(y) - c,$$

it is optimal to reject x.

PROOF: (i) If $x \geqslant \Sigma P_i x_i$, then, using Corollary 1, we have

$$x \geqslant V(\mathbf{P}) - c,$$

and (i) is established.

(ii) Suppose the present state is $(x, \mathbf{P})$, and consider the policy that accepts the first offer greater than x. The expected return from this policy is

$$\Sigma\, P_i \left[\int_x^\infty \frac{y dF_i\,(y)}{1 - F_i\,(x)} - \frac{c}{1 - F_i\,(x)}\right],$$

which follows by noting that, given that the distribution is F_i, the expected number of additional offers that will be made until one is accepted is $1/[1 - F_i\,(x)]$. Clearly, if x is less than this value, then it cannot be optimal to accept the present offer of x.

(iii) The proof of (iii) is similar to that of (ii) in that it considers the return when in state $(x, \mathbf{P})$ if you accept the next offer, and it notes that if this return is greater than x then x should clearly not be accepted.

REMARK: It follows from part (ii) of the above proposition that if $x < \min(x_1, \ldots, x_n)$ then it is always optimal to reject x.

Let us now consider the special case where there are only two possible distributions, i.e., F_1 and F_2, and suppose $x_1 \leqslant x_2$. In this case the state can be represented as the pair (x, P) where x is the present offer and P is the present probability (given all information, including, x, accumulated up to this point) that F_2 is the true distribution. In this case we have

THEOREM 2: $V(P)$ is an increasing function of P, $0 \leqslant P \leqslant 1$.

PROOF: Since $V(P)$ is a convex function of P (Proposition 4), the result would follow if we could show that

$$V(0) \leqslant V(P) \text{ for all } 0 \leqslant P \leqslant 1.$$

Now $V(0) = x_{F_1} + c \equiv x_1 + c$. Also, as it is always optimal to reject an offer less than $\min(x_1, x_2) = x_1$, it follows from the optimality equation that

$$V(P) - c \geqslant x_1 \text{ for all } P,$$

which proves the result.

Thus, when $n = 2$ and $x_1 \leqslant x_2$, it is optimal to accept the offer when in state (x, P) if and only if $x \geqslant h(P)$, where $h(P) \equiv V(P) - c$ is an increasing convex function of P with $h(0) = x_1$, $h(1) = x_2$. Furthermore, bounds on $h(P)$ are given by Proposition 5.

REMARK: There does not appear to be an analogue to Theorem 2 when there are more than 2 possible distributions. For instance, suppose that the distributions $F_1, F_2, \ldots, F_n$ are stochastically increasing in the sense that $F_i(t)$ is nonincreasing in i for each t. If we define the probability vector $\mathbf{P}$ to be greater than or equal to the probability vector $\mathbf{Q}$, written $\mathbf{P} \geqslant \mathbf{Q}$, if

$$\sum_1^j P_i \leqslant \sum_1^j Q_i \text{ for each } j = 1, \ldots n,$$

then we might hope to prove that $V(\mathbf{P}) \geqslant V(\mathbf{Q})$. However, this need not be the case, as is indicated by the following example. Suppose F_1 puts all its weight on the value 0.9, F_2 puts all its weight on the value 1, and F_3 is the distribution of a random variable that takes on the value 1 with probability 0.99 and $(10)^6$ with probability 0.01, and suppose $c = 1$. Now, $\mathbf{P} \equiv (0, 0.9, 0.1) \geqslant \mathbf{Q} \equiv (0.9, 0, 0.1)$, but it turns out that $V_{(\mathbf{P})} < V_{(\mathbf{Q})}$, the reason being that under $\mathbf{Q}$ it only takes a single observation to determine the true F_i, whereas this is not so under $\mathbf{P}$.

4. INDEPENDENT AND IDENTICALLY DISTRIBUTED OFFERS FROM AN UNKNOWN DISTRIBUTION WITH RECALL OF PAST OFFERS

In the previous section we assumed that once an offer was rejected by the decision maker then that offer immediately disappears. In this section, however, we consider the same model as in Section 3 but with the exception that an offer remains good indefinitely and may be accepted at any time.

It turns out that, when the distribution of offers is known, then the optimal policy in this case is identical to the one in which recalling past offers is not allowed. That is, the optimal policy is to accept the first offer that is at least as large as x_F, and the expected return under the optimal policy is $x_F + c$, when x_F is as defined in Section 2.

Consider now the case in which the distribution of offers is one of the distributions $F_1, \ldots, F_n$, where the F_i is chosen according to some initial probability vector. The state of the system at any time can be defined by $(m, \mathbf{P})$, where m is the maximum offer that has been received up to that time and $\mathbf{P}$ is the posterior probability vector (given all offers up to that time, including any just made) of the true distribution. The optimality equation takes the form

$$V(m, \mathbf{P}) = \max \{m, \sum_i P_i \left[\int_0^m V(m, T_y \mathbf{P})\, dF_i(y) + \int_m^\infty V(y, T_y \mathbf{P})\, dF_i(y)\right] - c\},$$

where

$$T_y \mathbf{P} = [(T_y \mathbf{P})_1, \ldots, (T_y \mathbf{P})_n],$$

and

$$(T_y \mathbf{P})_j = \frac{P_j\, dF_j\,(y)}{\Sigma P_i\, dF_i\,(y)}.$$

While it follows from its definition that $V(m, \mathbf{P})$ is an increasing function of m for fixed $\mathbf{P}$, it is not immediately evident from the optimality equation that, if the offer m is accepted when in state $(m, \mathbf{P})$, then the offer m^1 is also accepted when in state $(m^1, \mathbf{P})$ whenever $m^1 \geqslant m$. We now prove this.

PROPOSITION 6: For fixed $\mathbf{P}$, $V(m, \mathbf{P}) - m$ is a nonincreasing function of m.

PROOF: Suppose $m_1 < m_2$. Note that the distribution of the sequence of future offers is the same no matter whether the initial state is $(m_1, \mathbf{P})$ or $(m_2, \mathbf{P})$, since it only depends on $(x, \mathbf{P})$ through $\mathbf{P}$. We can then conclude that if the initial state is $(m_1, \mathbf{P})$ then, by following throughout the optimal policy for the initial state $(m_2, \mathbf{P})$, our return when we stop is with in $m_2 - m_1$ of what it would have been if the initial state were really $(m_2, \mathbf{P})$. Therefore,

$$V(m, \mathbf{P}) \leqslant \Sigma P_i\, V(m, e_i) = \Sigma P_i \max\{m, x_i\}.$$

COROLLARY 2: If it is optimal to accept m_1 when in state $(m_1, \mathbf{P})$, then it is optimal to accept m_2 when in state $(m_2, \mathbf{P})$ whenever $m_2 \geqslant m_1$.

PROOF: If $V(m_1, \mathbf{P}) = m_1$, then from Proposition 6

$$V(m_2, \mathbf{P}) - m_2 \leqslant 0.$$

This implies, from the optimality equation, that

$$V(m_2, \mathbf{P}) = m_2.$$

PROPOSITION 7: For fixed m, $V(m, \mathbf{P})$ is a convex function of $\mathbf{P}$.

PROOF: The proof is identical to the proof of Proposition 4 in Section 3.

COROLLARY 3: $V(m, \mathbf{P}) \leqslant \Sigma P_i \max(m, x_i)$,
where $x_i \equiv x_{F_i}$.

PROOF: If we let e_i be the vector of zeros with a one in the ith place then

$$V(m, e_i) = \begin{cases} m \text{ if } m > x_i \\ x_i \text{ if } m < x_i \end{cases}.$$

Hence, from convexity

$$V(m, \mathbf{P}) \leqslant \Sigma P_i\, V(m, e_i) = \sum_i P_i \max\{m, x_i\}.$$

PROPOSITION 8: If the present state is $(m, \mathbf{P})$ then

(i) if $m > \Sigma P_i \max(m, x_i)$ then it is optimal to accept m.

(ii) if

$$m < \Sigma P_i \left[\frac{\int_m^\infty y dF_i(y) - c}{1 - F_i(m)} \right]$$

then it is optimal to look at another offer.

(iii) if

$$m < \Sigma P_i \left[m F_i(m) + \int_m^\infty y dF_i(y) \right] - c$$

then it is optimal to look at another offer.

PROOF: Part (i) follows directly from Corollary 3, while the proofs of parts (ii) and (iii) are identical to the corresponding results of Proposition 5 in Section 3.

Suppose now that $n = 2$ and $x_1 \leqslant x_2$. In this case we represent the state by (m, P) when P is the posterior probability that F_2 is the true distribution.

THEOREM 3: $V(m, P)$ is increasing in P for fixed m.

PROOF: As in the corresponding proof of the previous section, we need to show that

$$V(m, 0) \leqslant V(m, P).$$

Now,

$$V(m, 0) = \max(m, x_1).$$

However,

$$V(m, P) \geqslant m,$$

and, as it follows from Part (ii) of Proposition 8 that it is never optimal to accept an offer less than x_1, we have

$$V(m, P) \geqslant x_1.$$

That is,

$$m > x_1 \rightarrow V(m, P) \geqslant x_1$$

$$m < x_1 \rightarrow V(m, P) = V(x_1, P) > x_1,$$

and the proof is complete.

Hence, when $n = 2$ and $x_1 \leqslant x_2$, it is optimal to accept m when in state (m, P) if and only if $m \geqslant m(P)$, where $m(P)$ is an increasing convex function of P with $m(0) = x_1$, $m(1) = x_2$.

REFERENCES

[1] Barlow, R., and F. Proschan, *Statistical Theory of Reliability and Life Testing Probability Models* (Holt, Rinehart & Winston, New York, 1977).

[2] Chow, Y.S., and H. Robbins, "A Martingale Systems Theorem and Applications," in *Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability* Vol. 1, pp. 93-104 (1961).

[3] DeGroot, M.H., "Some Problems of Optimal Stopping," Journal of the Royal Statistical Society, Series B *30*, No. 1, 108-122 (1968).

COMPUTATIONAL RESULTS WITH A BRANCH-AND-BOUND ALGORITHM FOR THE GENERAL KNAPSACK PROBLEM

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ABSTRACT

In this paper, a branch-and-bound procedure is presented for treating the general knapsack problem. The fundamental notion of the procedure involves a variation of traditional branching strategies as well as the incorporation of penalties in order to improve bounds. Substantial computational experience has been obtained, the results of which would indicate the feasibility of the procedure for problems of large size.

INTRODUCTION

The general knapsack problem considered in this study is

$$P: \quad \text{Maximize} \quad Z = \sum_{j=1}^{n} c_j x_j \tag{1}$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_j x_j \leqslant b, \tag{2}$$

$$0 \leqslant x_j \leqslant u_j, \tag{3}$$

and

$$x_j \text{ integer.} \tag{4}$$

Without loss of generality, we can assume c_j, a_j, and b to be integers and, further, that a_j and b are nonnegative scalars [6]. In addition, we assume that the indices are ordered such that $c_1/a_1 \geqslant c_2/a_2 \geqslant \ldots \geqslant c_n/a_n$.

THE APPROACH

The general strategy we adopt in this work is the branch-and-bound approach of Dakin [3] which, itself, is essentially a modification of the landmark algorithm of Land and Doig [7].

Also, the strategy we pursue incorporates the suggestion of Beale and Small [2] to implement the concept of penalities in order to improve the bounds used by Dakin. The reader will find the discussion in Sections 4.1-4.3 of Taha [13] and in Geoffrion and Marsten [5] useful in understanding the terminology and concepts used in this paper.

Node Selection

Given a list of subproblems (nodes) with integer variables, we need to select one for further exploration. We make such a decision by selecting the subproblem which possesses a nonintegral optimal solution and has the largest upper bound. Here, the upper bound $\hat{Z}$ for a problem CP is given by

$$\hat{Z} = [Z(\overline{CP}) - P_T], \tag{5}$$

where $Z(\overline{CP})$ is the optimal objective value of problem $\overline{CP}$, which is a linear program obtained from problem CP by relaxing the integrality constraints, [k] denotes the largest integer not exceeding k, and P_T is the penalty for imposing integrality. The form of P_T has been derived by Tomlin [14] and can be given in explicit form as follows: Let x^* define the optimal solution to the relaxed problem and let x_p^* be the fractional-valued variable. The coefficients in the optimal tableau obtained by using the simplex procedure with upper-bounded variables are given by

$$\bar{a}_i = \begin{cases} -a_i/a_p & \text{if } x_i^* = 1, \\ a_i/a_p & \text{otherwise;} \end{cases}$$

$$\bar{c}_i = \begin{cases} c_i + c_p^* \bar{a}_i & \text{if } x_i^* = 1, \\ -c_i + c_p^* \bar{a}_i & \text{otherwise,} \end{cases}$$

and $\bar{b} = x_p^*$. The penalty P_T is then given by

$$P_T = \underset{i \neq p}{\text{minimum}} \begin{cases} \bar{b}\ \bar{c}_i/f_i, & f_i \leqslant \bar{b}, \\ (1-\bar{b})\bar{c}_i/(1-f_i), & f_i \geqslant \bar{b}, \end{cases}$$

where $f_i = \bar{a}_i - n_i$ for some integer n_i and $0 \leqslant f_i < 1$.

Fathoming

Suppose Z^* is the value of the current best solution to the problem (the incumbent solution). For a node selected with associated problem CP, the node is considered fathomed if its relaxed linear program $\overline{CP}$ is infeasible, has the upper bound $Z \leqslant Z^*$, or has an integral optimal solution. In the last instance, if $Z > Z^*$, then a new incumbent has been found.

Branching

Suppose a node and the corresponding problem CP are selected for exploration. If the fathoming test fails, it is necessary to create two new branches (subproblems). This is done by choosing a branching variable x_v and suitably changing its lower or upper bound. The selection process is designed so that the bound computed from (5) is as small as possible for one of the problems, and hence can be eliminated early.

During the procedure, the values of certain variables will be fixed. We will call the remaining variables *free variables*. Selection of an index v from among the free variables is central to the branching process, and its determination can be summarized by the following rule. Let v_1 be the smallest index of the free variable and v_2 the largest index of the free variable in problem *CP*. Further, let us consider two problems, one of which has the upper bound on variable v_1 decreased by one and the other of which has the lower bound on variable v_2 incremented by one. Let $\hat{Z}_1$ and $\hat{Z}_2$ be the bounds obtained by (5) for these problems. If $\hat{Z}_1 \geqslant \hat{Z}_2$, we let $v = v_2$. On the other hand, if $\hat{Z}_1 < \hat{Z}_2$, we let $v = v_1$. Note that if q is the index of the fractional-valued variable in problem *CP* and if $v_1 = q$, then $v = v_2$. Likewise, if $v_2 = q$ then $v = v_1$. The process would detect a terminal node on the branch if $v_1 = v_2 = q$.

Once the index v is selected, we create two problems CP_1 and CP_2 with integer variables. We will denote the problems obtained by relaxing the integrality constraints by $\overline{CP}_1$ and $\overline{CP}_2$ respectively. If x_v^* is the optimal value of variable x_v in problem $\overline{CP}$, the relaxation of *CP*, we obtain problem CP_1 by adding the constraint $x_v = x_v^*$ to problem *CP*, so that x_v is now *fixed*. Hence, the solution to problem $\overline{CP}$ continues to be optimal to problem $\overline{CP}_1$. To create the second subproblem CP_2, the following rule is adopted:

(i) If $v < q$ (so that $v = v_1$), add to problem *CP* the constraint $x_v \leqslant u_v'$, where $u_v' = x_v^* - 1$. In this case, x_v is at its current upper bound in problem $\overline{CP}$, and in problem $\overline{CP}_2$ its optimal value will be $x_v^* - 1$.

(ii) If $v > q$ (so that $v = v_2$), add to problem *CP* the constraint $x_v \geqslant l_v'$, where $l_v' = x_v^* + 1$. In this case, x_v is at its current lower bound in problem $\overline{CP}$, and in problem $\overline{CP}_2$ its optimal value will be $x_v^* + 1$.

Initial Solution

Woolsey and Swanson [15] have given a procedure, referred to as the slippery algorithm, which yields good approximate solutions to a 0-1 knapsack problem. A slight modification of this scheme to treat the general problem is self-evident, and it was used to obtain starting solutions in this study. We point out, however, that the procedure in [15] is based upon the notion of determining a "greedy" solution (see Magazine, et al. [8]).

COMPUTATIONAL ASPECTS

Once the basic notions of the branch-and-bound scheme are in hand, specific computational savings can be highlighted. For example, suppose $\hat{Z}_1$ and $\hat{Z}_2$ are computed for problem *CP* in order to select a branching variable v. If $v = v_1$, then x_v is fixed in CP_1. To select a branching variable for CP_1, the free variable with the largest index is still v_2. Hence, the bound $\hat{Z}_2$ is still valid if v_2 is larger than the index of the fractional variable in the solution CP_1. A similar statement can be made if $v = v_2$.

The reader may also note that the solution to problem $\overline{CP}$ can be used in solving problem $\overline{CP}_2$. Recall that $\overline{CP}$ and $\overline{CP}_2$ differ only by an upper or a lower bound on one variable. Suppose $v = v_1$, so that if u'_v is the upper bound on x_v in problem *CP*, we add the constraint $x_v \leqslant u'_v - 1$ to get problem CP_2. Then the optimal values of variables x_j for $j < v$ are the same in problems $\overline{CP}$ and $\overline{CP}_2$. For variable x_v the optimal value is $u'_v - 1$ in problem $\overline{CP}_2$. A similar statement can be made when $v = v_2$.

For testing, the branch-and-bound procedure discussed above was coded in FORTRAN and numerous problems were run on a Cyber 74 system. The parameters c_j and a_j were randomly generated from the uniform distributions [1, 50], [1, 100], and [1, 999]. Within each of these classes, the values of u_j were randomly generated over [1,5] and [1,10]. The only exception was in the case of problems with n of 1000, 1500, and 2000, in which a reduced set of experiments was considered. In all problem classes, the knapsack size b was randomly generated over $[\alpha/3, 2\alpha/3]$, where $\alpha = \sum_{j=1}^{n} a_j u_j$. All computational experience is reported in Tables 1-3. All times are in *c.p.u.* seconds and do not include the time for ordering ratios c_j/a_j. That such times are not included is consistent with other studies dealing with knapsack problems.

Five random problems in each classification were run to determine the mean values in tables. Certainly, a rigorous analysis regarding problem behavior relative to size and other parametric variation would not be possible with such sample sizes. Some trends are suggested, however, by the data generated. For example, it would seem that a more marked impact on computing effort is realized by increasing the ranges on parameters c_j and a_j than by, say, increasing the problem size within a given range. Note, for example, that the mean time for solving 1000 variable problems was slightly less than that for 600 variable problems where the latter possessed c_j and a_j values randomly generated over [1, 999], and the former, over [1, 50] (see Tables 1 and 3). In both cases $u_j \leqslant 5$.

An expected result of the experiment also occurs when the bound u_j is increased. In most cases, the increase fosters a similar increase in computing effort. Of particular interest here is the number of nodes generated. Although only two upper-bound values of u_j were considered in the experiment reported in the tables, numerous other problems were solved with bound values as great as 30, in which case the increase in computing effort was generally observed to be substantial compared to those cases reported.

FUTURE RESEARCH

Diagnostic analysis relative to the composition of "difficult" problems would be an obvious area for further study. Throughout this research, it was clear that substantial variation in computational effort could occur relative to changes in problem parameters. Frequently, such parametric variation was subtle. For example, it would appear from the results attained in this study that a critical dimension in problem difficulty is the range of parameters a_j and c_j as contrasted to an increase in problem size n. In addition, it was found that by judicious choices of differing distributions for a_j and c_j values particularly troublesome problems could be created. Note that in the experiment reported values of a_j and c_j were always generated from the same distribution.

In conclusion, it may be of interest to note that the thought underlying the branching strategy suggested in this work is somewhat akin (although independently developed) to that expressed by Balas and Zemel [1] as well as others in the case of the 0-1 problem. An interesting element in these works is the suggestion that a "core problem" exists which involves only a subset of variables whose values are of significance. We have not attempted to address this notion in the current work but its analysis and/or substantiation would seem to hold merit. This is especially important in the light of particularly impressive computational results arising from the current work on 0-1 problems [1, 9, 12, 17].

TABLE 1. *Summary of Computational Experience,* $a_j, c_j \in [1, 50]$

Problem Size, n	$u_j \in [1, 5]$					$u_j \in [1, 10]$				
	mean time	max. time	min. time	mean* nodes	max. nodes	mean time	max. time	min. time	mean* nodes	max. nodes
500	0.510	1.276	0.047	199.4	504	0.919	1.732	0.019	302.2	508
600	1.761	2.156	1.193	453.0	627	1.217	2.622	0.056	394.8	792
700	0.567	2.634	0.020	139.8	699	1.403	2.626	0.026	406.8	701
800	1.758	3.590	0.027	457.4	838	2.424	5.097	0.022	543.2	1086
900	3.538	6.156	0.022	764.8	1179	0.708	3.703	0.023	175.8	879
1000	4.302	5.214	3.839	964.8	1026	1.834	5.408	0.040	387.6	1080
1500	1.464	6.972	0.053	248.6	1243	6.994	12.080	0.040	1072.0	1501
2000	6.405	16.917	0.078	777.2	2021	2.429	11.730	0.050	326.2	1631

*Average value of the maximum number of nodes ever maintained in storage.

TABLE 2. *Summary of Computational Experience,* $a_j, c_j \in [1, 100]$

Problem Size, n	$u_j \in [1, 5]$					$u_j \in [1, 10]$				
	mean time	max. time	min. time	mean nodes	max. nodes	mean time	max. time	min. time	mean nodes	max. nodes
500	1.868	3.744	0.018	315.0	547	2.090	2.870	1.125	560.2	755
600	1.860	2.667	0.016	498.4	640	2.223	2.369	1.944	504.6	624
700	3.002	4.907	0.060	630.6	917	4.035	6.805	0.024	627.0	830
800	4.170	9.115	0.096	736.0	1088	3.505	5.519	0.095	687.0	891
900	2.176	6.590	0.032	402.8	1005	6.201	7.792	3.743	962.0	1057

TABLE 3. *Summary of Computational Experience,* $a_j, c_j \in [1, 999]$

Problem Size, n	$u_j \in [1, 5]$					$u_j \in [1, 10]$				
	mean time	max. time	min. time	mean nodes	max. nodes	mean time	max. time	min. time	mean nodes	max. nodes
500	7.128	18.072	2.129	698.8	1072	9.415	19.152	1.930	830.8	1154
600	4.838	12.138	1.782	723.6	1050	5.653	14.479	3.075	715.6	974
700	5.220	6.613	3.971	767.4	806	6.825	10.258	3.128	861.6	1046
800	10.151	19.922	3.954	973.4	1269	5.813	9.069	4.072	931.8	1243
900	21.421	38.993	5.665	1260.4	1568	12.734	24.678	7.895	1099.8	1267

BIBLIOGRAPHY

[1] Balas, E., and E. Zemel, "Solving Large Knapsack Problems," presented at the Joint National Meeting of ORSA/TIMS, Miami, Florida (November, 1976).

[2] Beale, E.M.L., and R.E. Small, "Mixed Integer Programming by a Branch-and-Bound Technique," *Proc. IFIP Congress, New York, May, 1965*, W.A. Kalenich, ed. (Spartan Books, Washington, D.C., 1965, pp. 450-451.

[3] Dakin, R.J., "A Tree-Search Algorithm for Mixed Integer Programming Problems," Computer Journal, **8**, pp. 250-255, (1965).

[4] Garfinkel, R.S., and G.L. Nemhauser, *Integer Programming* (Wiley, New York, 1972).

[5] Geoffrion, A.M., and R.E. Marsten, "Integer Programming: A Framework and State-of-the-Art Survey," Management Science, **18**, 465-491 (1972).

[6] Glover, F., "A Multiphase-Dual Algorithm for the Zero-One Integer Programming Problem," Operations Research, **13**, 879-919 (1965).

[7] Land, A.H., and A.G. Doig, "An Automatic Method for Solving Discrete Programming Problems," Econometrica, **28**, 497-520, (1960).

[8] Magazine, M.J., G.L. Nemhauser, and L.E. Trotter, Jr., "When the Greedy Solution Solves a Class of Knapsack Problems," Operations Research, **23**, 207-217 (1975).

[9] Nauss, R.M., "An Efficient Algorithm for the 0-1 Knapsack Problem," Management Science, **23**, 27-31 (1976).

[10] Salkin, H.M., *Integer Programming* (Addison-Wesley, Reading, Massachusetts, 1975).

[11] Salkin, H.M., and C.A. deKlayver, "The Knapsack Problem: A Survey," Naval Research Logistics Quarterly, **22** 127-144 (1975).

[12] Suhl, U., "An Algorithm and Efficient Data Structures for the Binary Knapsack Problem," Arbeitspapier Nr. 20/1976, Fachbereich Wirtschaftswissenschaft, Freie Universität Berlin (1977).

[13] Taha, H.A., *Integer Programming: Theory, Applications and Computations* (Academic Press, New York, 1975).

[14] Tomlin, J.A., "An Improved Branch-and-Bound Method for Integer Programming," Operations Research, **19**, 1070-1074, (1971).

[15] Woolsey, R.E.D., and H.S. Swanson, *Operations Research for Immediate Application — A Quick and Dirty Manual*, (Harper and Row, New York, 1975).

[16] Zionts, S., *Linear and Integer Programming* (Prentice-Hall, Englewood Cliffs, New Jersey, 1973).

[17] Zoltners, A.A., "A Direct Descent Binary Knapsack Algorithm," Working Paper 75-31, School of Business Administration, University of Massachusetts, Amherst (1975).

M/M/1 QUEUES WITH INTERDEPENDENT ARRIVAL AND SERVICE PROCESSES

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ABSTRACT

We study via simulation an M/M/1 queueing system with the assumption that a customer's service time and the interarrival interval separating his arrival from that of his predecessor are correlated random variables having a bivariate exponential distribution. We show that positive correlation reduces the mean and variance of the total waiting time and that negative correlation has the opposite effect. By using spectral analysis and a nonparametric test applied to the sample power spectra associated with certain simulated waiting times we show the effect to be statistically significant.

INTRODUCTION

In M/M/1 queues it is usually assumed that a customer's service time is independent of the time interval separating his arrival from that of his predecessor. Designating the difference in arrival epochs of customers C_{n-1} and C_n by T_n and the service time of customer C_n by S_n we depart from the usual assumption of independence and take the pair (T_n, S_n) to be a bivariate exponential random variable (r.v.) with correlation coefficient ρ. For $\rho = 0$ we have the simple queueing system commonly studied.

Correlated queues of this nature have received very little attention in the literature. Bhat [2] describes five different classes of single-server queues which are closer to real systems than they would be with assumptions of independence, and he states that more work needs to be done in these areas. One of the classes is of systems with interdependent arrival and service processes, as assumed here for the pair (T_n, S_n).

Conolly [3] gives the waiting-time distribution and its moments for the unusual single-server system where arrivals are from a Poisson process and the ratio S_n/T_n is constant for all n. It is also shown that this pattern of server behavior results in a drastic reduction in the mean and variance of the waiting times as compared with the conventional M/M/1 system. Conolly refers to this type of system as self-regulating, and other results are given by him and Hadidi [6,7]. We study herein, via simulation, this type of system where the dependence between the service time and the interarrival interval is probabilistic. Since this note was first written,

Conolly and Choo [4,5] have obtained a substantial number of results of an analytical nature regarding these self-regulating systems. Their results make possible an exact analysis of many aspects of the queueing system we introduce in this note.

2. A BIVARIATE EXPONENTIAL DISTRIBUTION

There are infinitely many ways of defining a bivariate exponential distribution with exponential margins; however, only a few such distributions have received any attention in the literature. Because of its many attractive mathematical properties, the most commonly used bivariate exponential distribution is the so-called Wicksell-Kibble distribution [11,15]. We drop the subscript n for now and take the density function for the bivariate random variable (T, S) with correlation $0 \leqslant \rho < 1$ to be

$$f(t,s) = (1-\rho)\lambda\mu e^{-\lambda t-\mu s} I_0\,[2(\rho\lambda\mu st)^{1/2}], \tag{1}$$

where $t \geqslant 0$, $s \geqslant 0$, and $I_0(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{(k!)^2}$ is the modified Bessel function of the first kind and order zero. The density (1) has mean vector

$$\begin{bmatrix}\mu_T\\ \mu_S\end{bmatrix} = \begin{bmatrix}(\lambda\rho')^{-1}\\ (\lambda\rho')^{-1}\end{bmatrix} \tag{2}$$

and covariance matrix

$$\Sigma = \begin{bmatrix}\sigma_T^2 & \sigma_{TS}\\ \sigma_{TS} & \sigma_S^2\end{bmatrix} = \begin{bmatrix}\mu_T^2 & \rho\mu_T\mu_S\\ \rho\mu_T\mu_S & \mu_S^2\end{bmatrix}, \tag{3}$$

where $\rho' = 1-\rho$. The marginal distributions of T and S are exponential with means μ_T and μ_S respectively. A generalization of (1) of Paulson [14] admits of correlation values $-0.25 \leqslant \rho < 1$. It is readily shown that the pair (T, S) with density (1) has a two-dimensional Laplace transform given by

$$\phi(u, v) = (1-\rho)\left[\left(1+\frac{u}{\lambda}\right)\left(1+\frac{v}{\mu}\right)-\rho\right]^{-1}, \tag{4}$$

which clearly reduces to the Laplace transform of a pair of independent exponential variates when $\rho = 0$. The density (1) does not admit of negative correlations, so that for the case $\rho < 0$ we use herein the generalization of Paulson [14]. The especially simple form of (4) makes it possible to obtain analytical results which with other choices of bivariate exponential distributions would be hard to come by.

Whereas there is only *one* exponential distribution in one dimension, there are *infinitely* many "potential" exponential distributions in higher dimensions. However, since we would naturally require that any bivariate exponential distribution have a rational Laplace transform (such as (4)), it does not seem possible that any other bivariate exponential distribution would be "much different" from (1) and hence the use of (1) in the study of self-regulating queueing systems should lead to generic results, i.e., system behavior for one distribution should typify behavior for the other "potential" distributions. This is likely to be true even for the singular Marshall-Olkin distribution [14].

We have restricted this note to considerations involving only the bivariate exponential distribution; consideration of other distributions seems patently merited since, as we shall see, system behavior is so drastically affected by correlation. A particularly useful generalization of the density (1) obtainable through the preservation (to higher dimensions) scheme of Paulson [14] has the Laplace transform

$$\phi(u, v) = (1-\rho)\left[\left(1+\frac{u}{\lambda}\right)^{\alpha}\left(1+\frac{v}{\mu}\right)^{\beta}-\rho\right]^{-1}, \tag{5}$$

where $\alpha>0$, $\beta>0$, $0\leqslant\rho<1$, and λ and μ are arrival and service parameters of the distribution. The density $g(t,s)$ corresponding to (5) reduces to (1) when $\alpha=\beta=1$. One margin of the density $g(t,s)$ is gamma (Erlang) with shape parameter α, the other margin is gamma with shape parameter β, and the variates T and S have correlation ρ. This density would be useful in the study of queueing systems other than the partially correlated M/M/1, for example, partially correlated $M/E_n/1$ or $E_n/M/1$. Here E_n denoted the Erlang (gamma) distribution of orders n.

The variates (T,S), given by (1) may be readily simulated. Let $\{X_{ij},\ j=1, 2, \ldots\}$ be independent identically distributed (i.i.d.) exponential random variables with means λ^{-1}, $i=1$, and μ^{-1}, $i=2$. Let N denote a geometric r.v. with density function

$$P[N=i]=\rho^{i-1}(1-\rho),\ i=1, 2, \ldots . \tag{6}$$

It follows that

$$(T,S)=\left(\sum_{j=1}^{N} X_{1j},\ \sum_{j=1}^{N} X_{2j}\right) \tag{7}$$

has the bivariate distribution (1) (see [12]). The simulation proceeds by our simulating N and then adding the number of i.i.d. exponentials according to (7). Other comments about this bivariate r.v. are given in Downton [8]. When $\rho<0$, simulation of exponential variates is described in [12].

3. SOME RESULTS

We assume that customers from an infinite population arrive at a single-server queue according to a Poisson process with rate λ; an unlimited queue is allowed and the service discipline is first come — first served with service rate μ and utilization $\nu=\lambda/\mu$. The sequence of pairs (T_n, S_n) is assumed to be i.i.d., that is, (T_n, S_n) is independent of (T_m, S_m) for every $m\neq n$ and all n. For customer C_n we assume that the r.v. has the distribution (1) when $0\leqslant\rho<1$ and Paulson's generalization of (1) when $-0.25<\rho<0$.

The quantity W_n is defined to be the total waiting time, queueing plus service, for customer C_n and it is clear that a recursive formula for W_{n+1} is

$$W_{n+1}=\begin{cases} W_n - T_{n+1} + S_{n+1}, & \text{if } T_{n+1} < W_n, \\ S_{n+1}, & \text{if } T_{n+1} \geqslant W_n \end{cases} \tag{8}$$

for all n. If $f(t,s)$ is the joint density of (T_n, S_n) given by (1), then the density of W_{n+1} is readily found to be (Conolly, personal communication, Conolly and Choo [4,5])

$$a_{n+1}(w)=\int_0^{\infty}\int_w^{\infty} a_n(w)\, f(t,s)\, dtds + \int_0^{\infty}\int_0^{w} a_n(w+t-s)\, f(t,s)\, dsdt \tag{9}$$

with corresponding Laplace transform

$$\alpha_{n+1}(z)=\alpha_1(z)\left[\frac{f(z)\alpha_n(z)-z\alpha_n f(z)}{f(z)-z}\right], \tag{10}$$

where

$$\alpha_1(z) = \frac{\mu\rho'}{z + \mu\rho'} \tag{11}$$

and

$$f(z) = \left(1 - \frac{\mu\rho}{z + \mu}\right). \tag{12}$$

It can be shown that $\lim_{n\to\infty} \alpha_n(z) = \alpha(z)$ where $\alpha(z)$ satisfies the functional equation

$$\alpha(z) = \frac{\mu\rho'(z + \mu)\alpha(f(z))}{(z + \mu\rho')(z + \mu\nu')} \tag{13}$$

which becomes on repeated iteration

$$\alpha(z) = \left(\frac{\mu\rho'}{z + \mu\rho'}\right)\left(\frac{z + \mu}{z + \mu\nu'}\right)\prod_{j=1}^{\infty}\left(\frac{\mu\rho'}{z_j + \mu\rho'}\right)\left(\frac{z_j + \mu}{z_j + \mu\nu'}\right), \tag{14}$$

where $z_j = f(z_{j-1})$, $j = 1, 2, \ldots, z_0 = z$, $\nu' = 1 - \nu$. The Laplace transform of the steady-state waiting-time distribution is $\tilde{\alpha}(z) = \alpha(z)/\alpha(0)$. Equations (13) and (14) contain complete information about the distribution of steady-state waiting time, but retrieval of this information in other than numerical form is more difficult than appearances would suggest. However, the mean and variance may be easily obtained by numerically differentiating $\tilde{\alpha}(z)$ as obtained from (14).

In the sequel we take, without loss of generality, $E[T_n] = \lambda = 1$ and $E[S_n] = 1/\mu = \nu < 1$. For T_n and S_n independent, as is normally assumed, the mean waiting time per customer, in steady state, is $\nu/(1 - \nu)$ [13]. At the other extreme, for $\rho = 1$ Conolly [3] gives the mean waiting time for the case $S_n = \nu T_n$ for all n. Our results are for other values of correlation. Next we use (8) to show simulated results.

In Fig. 1 we show how nonzero correlation affects the mean waiting time for $\nu = 0.70$. For zero correlation the expected waiting time in steady state is 2.333, and we see that the simulated result after 25,000 service completions is in close agreement (2.323). Conolly shows for his system ($\rho = 1$) that the expected waiting time for this value of ν is 1.427, and again the agreement is very good (1.421). From the figure it is clear that each graph tends to stabilize for increasing n in accordance with the law of large numbers. Thus, for positive correlation we have a benefit in system performance in that the mean waiting time decreases, and for negative correlation the process is degraded. (Starting conditions for the simulations were based on pilot runs.)

Figure 2 shows the ratio of mean waiting time in the system, for various values of ρ, to the expected waiting time in the system with $\rho = 0$, we see that the effect of nonzero correlation is greatest for large utilizations. The solid lines depict a smooth fit to the actual data. Sampling variation, of course, precludes the possibility of obtaining such a smooth fit without extremely long runs or extensive replications.

Figure 3 shows the standard deviation of the simulated waiting-time process for several values of the correlation coefficient ρ and for utilization equal to 0.7. For zero correlation the expected standard deviation in steady state is 2.333; the simulated value is 2.023. Conolly shows for $\rho = 1$ that the expected standard deviation is 0.783; the simulated value is 0.779.

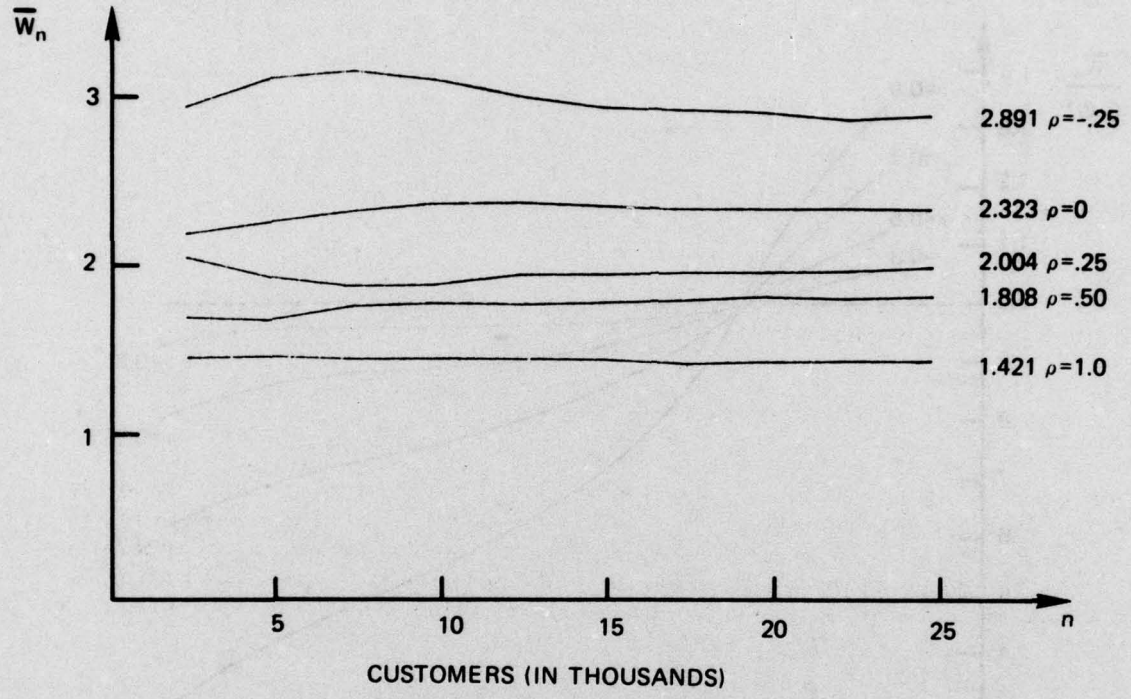

FIGURE 1. Mean waiting times, $\nu = 0.7$

The mean and standard deviation of waiting time in the system can be computed from $\alpha(z)/\alpha(0)$ as given by (14). The agreement between simulated values and exact numerical values is reasonably good in all cases. The greatest variability occurs for low and negative values of correlation. The least variability is evident for high values of correlation. There is a spectacular reduction in mean waiting time for high utilization and high correlation, but the reduction in the standard deviation is even greater. In any further simulations of this type, special care must be taken with low values of correlation, as is indicated by comparison of our simulated results with the exact results.

When we initiated this research the question of whether correlation between interarrival and service times had any effect on system behavior was moot. Exact expressions would have been ideal, but they were, unfortunately, lacking. The resolution of the question was effected through the use of spectral analytic techniques. Conolly and Choo [4,5] have since succeeded in providing an elegant and broader analysis of the partially correlated queueing system we have considered here.

4. SPECTRAL ANALYSIS OF $\{W_n\}$

In this section we review briefly the theory of spectral analysis, show sample power spectra of $\{W_n\}$ for different values of ρ, and, finally, apply a nonparametric test to the ratio of certain estimated power spectra.

Several authors give complete accounts of the theory and applications of spectral analysis; e.g., see Anderson [1], and Jenkins [10]. Fishman and Kiviat's [9] paper on the analysis of simulation generated time series is also of direct interest. The application of spectral analytic

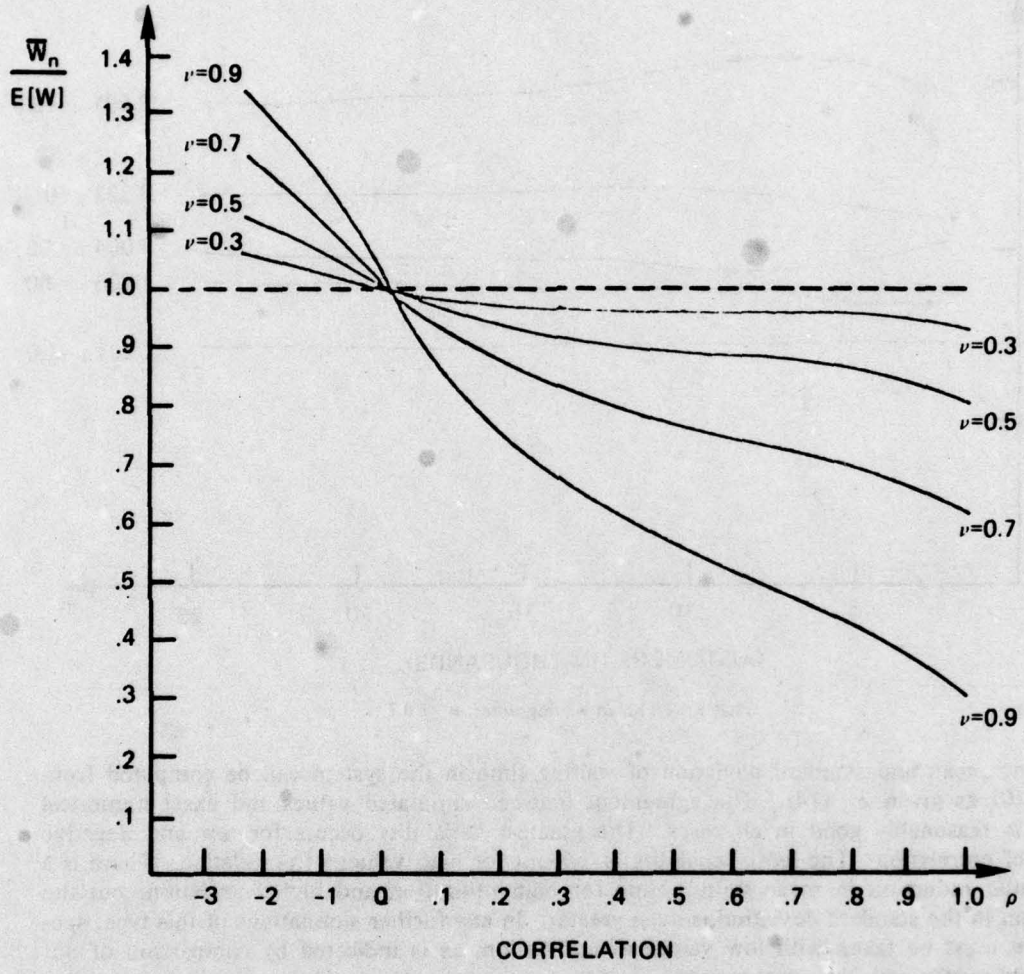

FIGURE 2. Ratio of mean waiting time at $\rho \neq 0$ to expected waiting time at $\rho = 0$

techniques to the waiting-time process for a tandem queueing system is discussed by Mitchell, et al. [12] and the application here is identical.

The $\{W_n, n = 1, 2, \ldots, N\}$ will be a realization of a stochastic process with mean μ and autocovariances γ_k, $k = 0, 1, \ldots$. A study of a time series in terms of its autocovariances is referred to as a time-domain analysis. Another type of analysis, spectral analysis, is concerned with the frequency content of the time series. The Fourier cosine transform of the autocovariances $\gamma_0, \gamma_1, \gamma_2, \ldots$ is called the power spectrum.

Denoting the power spectrum by $f(\omega)$, we can write

$$f(\omega) = 2[\gamma_0 + 2 \sum_{k=1}^{\infty} \gamma_k \cos 2\pi\omega k], \quad 0 \leq \omega \leq \frac{1}{2}, \tag{15}$$

and we typically estimate $f(\omega)$ with the truncated estimate

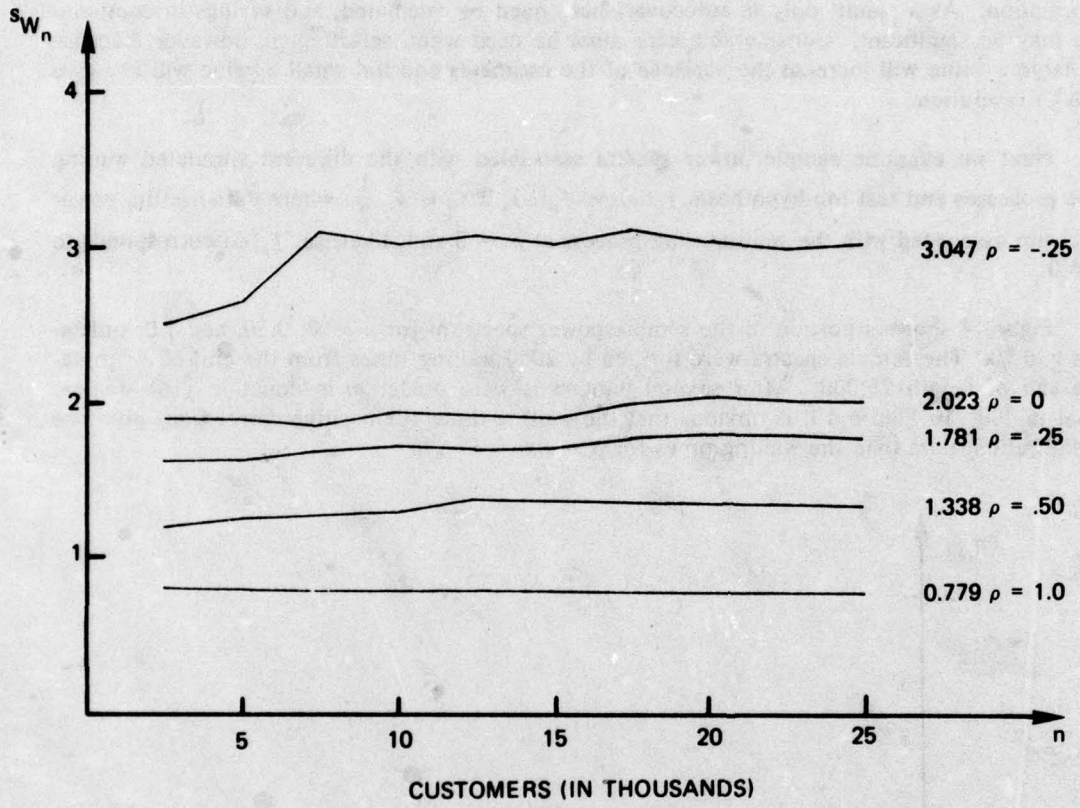

FIGURE 3. Standard deviation of waiting times, $\nu = 0.7$

$$\hat{f}(\omega_j) = 2[\lambda_0 c_0 + 2 \sum_{k=1}^{m} \lambda_k c_k \cos 2\pi \omega_j k], \tag{16}$$

where $\omega_j = j/(2m)$, $j = 0, 1, 2, \ldots, m$, the weights λ_k, $k = 0, 1, 2, \ldots, m$, form a so-called lag window, and the c_k, $k = 0, 1, 2, \ldots, m$, are the sample estimates of the autocovariances. We choose the Blackman-Tukey "hamming" window given by

$$\lambda_k = 0.54 + 0.46 \cos \pi k/m, \quad k = 0, 1, 2, \ldots, m, \tag{17}$$

and for the estimates of the autocoveriances we use

$$c_k = \frac{1}{N} \sum_{n=1}^{N-k} (W_n - \overline{W})(W_{n+k} - \overline{W}), \tag{18}$$

where

$$\overline{W} = \frac{1}{N} \sum_{n=1}^{N} W_n \tag{19}$$

and N is the number of observations in the time series $\{W_n\}$. In (16), the sample autocovariances $c_{m+1}, c_{m+2}, \ldots$ are omitted since, for m sufficiently large, they should contribute little

information. As a result, only m autocovariances need be calculated, and savings in computation may be significant. Considerable care must be used when selecting m, however, because too large a value will increase the variance of the estimates and too small a value will not give enough resolution.

Next we examine sample power spectra associated with the different simulated waiting time processes and test the hypothesis $f_0(\omega) = f_\rho(\omega)$, $0 \leqslant \omega \leqslant \frac{1}{2}$, where $f_0(\omega)$ is the power spectrum associated with the waiting time process at $\rho = 0$ and, likewise, $f_\rho(\omega)$ corresponds to $\rho \neq 0$.

Figure 4 shows a portion of the sample power spectrum for $\rho = 0$, 0.50, and 1.0; utilization is 0.70. The sample spectra were formed by 2000 waiting times from the end of a simulation run of length 25,000. After several pilot runs were made, m in equation (16) was set equal to 400. In Figure 4 it is obvious that the waiting times for positive correlations give rise to different spectra than the waiting times for $\rho = 0$.

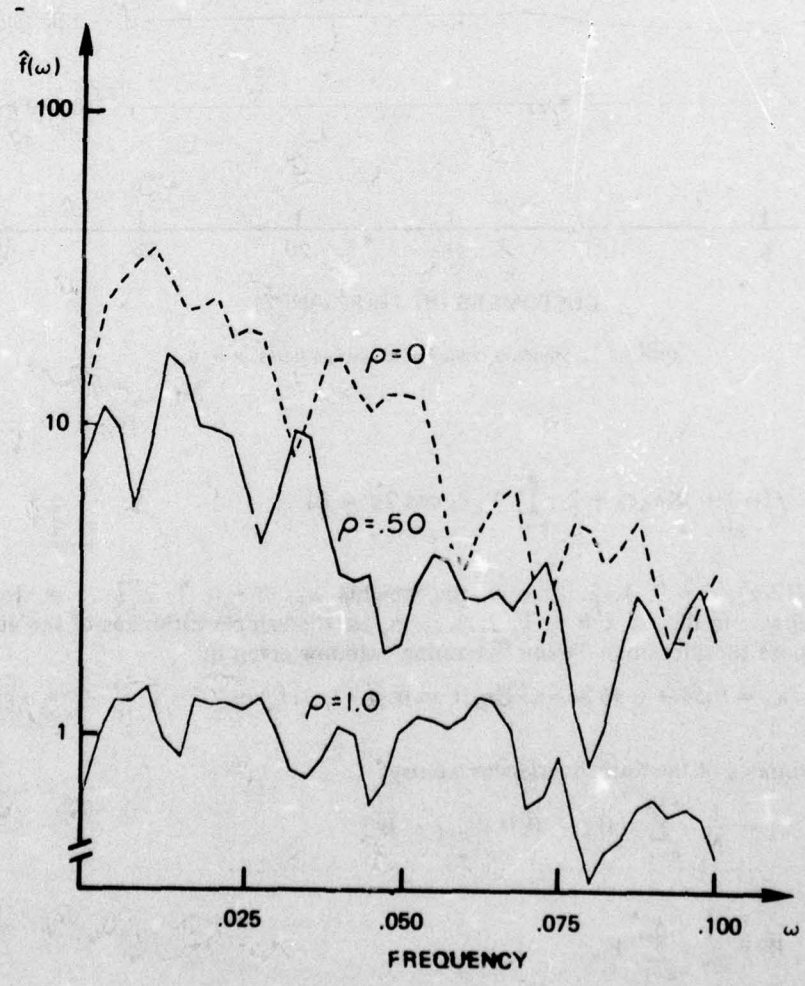

FIGURE 4. A portion of waiting time spectra, $\nu = 0.7$

Since the integral of the power spectrum measures the variance of the process [1,10] and the area under the sample spectrum should be indicative of the sample variance, the illustrated graphs in Figure 4 show that the variance of the waiting times decreases with positive correlation. In fact, the corresponding simulated waiting-time series have variances σ as follows: $\rho = 0$, $\sigma = 4.763$; $\rho = 0.50$, $\sigma = 2.141$; and $\rho = 1.0$, $\sigma = 0.514$. (The expected variance for $\rho = 0$ is 5.444 [13] and Conolly's model for $\rho = 1.0$ gives rise to a variance of 0.613.) Coupled with the simulated result that $\rho = -0.25$ leads to a variance of 10.451, we see that the effect of positive correlation is to reduce the variance of the waiting-time process and that negative correlation causes an increase.

The 401 spectral estimates of the power spectrum in (16) are not independent, and so we employ the notion of equivalent independent estimates. As developed in [10,12], we take spectral estimates at the frequencies $j/(2m)$, $j = 1, 4, 7, \ldots$, to be approximately independent and regard the ratio $\hat{f}_0(\omega)/\hat{f}_\rho(\omega)$ to be a Bernoulli trial (greater than unity or less than unity). Under a null hypothesis of homogeneity of the two spectra $f_0(\omega)$ and $f_\rho(\omega)$, we can take as a test statistic the number of ratios less than unity.

Figure 5 shows the ratio for $\rho = 0.5$ and $\nu = 0.7$. In the figure, 22 of the 134 approximately independent ratios are less than unity, which is very strong evidence that the null hypothesis is false; for other values of ρ the number of ratios less than unity are: 87 for $\rho = -0.25$, 43 for $\rho = 0.25$, and none for $\rho = 1.0$. Additionally, we applied the test to $f_{0.25}(\omega)$ and $f_{0.50}(\omega)$ and found 38 of the 134 approximately independent ratios to be less than unity. We reject the implied null hypothesis in all cases and conclude that the waiting-time process as a function of ρ leads to different power spectra. Similar results for other values of utilization and correlation obtain.

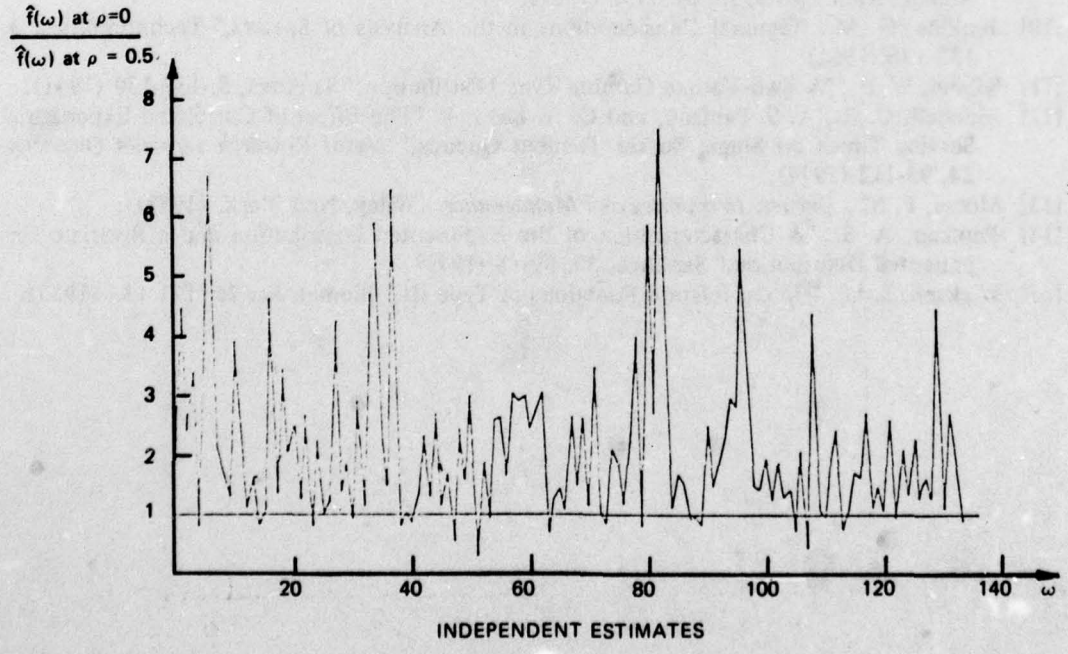

FIGURE 5. Ratio of sample power spectral estimates, $\nu = 0.7$

The investigation into the behavior of correlated queues is in its infancy. Even though an analytical analysis of this particular partially correlated queue has been obtained, it is highly probable that behavior of different queueing systems will be discovered, as here, by simulation.

ACKNOWLEDGMENT

We are grateful to Professor B. W. Conolly for allowing us to make use of as-yet unpublished results on exact system behavior and his helpful comments and suggestions.

REFERENCES

[1] Anderson, T. W., *The Statistical Analysis of Time Series*, (Wiley, New York, 1971).
[2] Bhat, U. N., "Queueing Systems with First-Order Dependence," *Opsearch, 6*, 1-24 (1969).
[3] Conolly, B. W., "The Waiting Time Process for a Certain Correlated Queue," Operations Research, *16*, 1006-1015 (1968).
[4] Conolly, B. W., and Q. H. Choo, "The Waiting Time Process for a Generalized Correlated Queue with Exponential Demand and Service," submitted for publication (1977).
[5] Conolly, B. W., and Q. H. Choo, "A Partially Correlated Queue," Research Report 77/1, Mathematics Department, Chelsea College, University of London, London, England.
[6] Conolly, B. W., and N. Hadidi, "A Correlated Queue," Journal of Applied Probability, *6*, 122-136 (1969).
[7] Conolly, B. W., and N. Hadidi, "A Comparison of the Operational Features of Conventional Queues with a Self-Regulating System," Applied Statistics, **18**, 41-53 (1974).
[8] Downton, F., "Bivariate Exponential Distributions in Reliability Theory," *Journal of the Royal Statistical Society*, Series B, **32**, 408-417 (1970).
[9] Fishman, G. S., and P. J. Kiviat, "The Analysis of Simulation-Generated Time Series," *Management Science*, **13**, 525-557 (1967).
[10] Jenkins, G. M., "General Considerations in the Analysis of Spectra," Technometrics, **3**, 133-136 (1961).
[11] Kibble, W. F., "A Two-Variate Gamma Type Distribution," Sankhyā, **5**, 137-150 (1941).
[12] Mitchell, C. R., A. S. Paulson, and C. A. Beswick, "The Effect of Correlated Exponential Service Times on Single Server Tandem Queues," *Naval Research Logistics Quarterly*, **24**, 95-112 (1977).
[13] Morse, P. M., *Queues, Inventories and Maintenance*, (Wiley, New York, (1958)).
[14] Paulson, A. S., "A Characterization of the Exponential Distribution and a Bivariate Exponential Distribution," Sankhyā, **35**, 69-78 (1973).
[15] Wicksell, S. D., "On Correlation Functions of Type III," Biometrika, **25**, 121-133 (1933).

CONFIDENCE INTERVALS RELATED TO SEQUENTIAL TEST FOR THE EXPONENTIAL DISTRIBUTION*

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ABSTRACT

One-sided sequential tests for the mean of an exponential distribution are proposed, and the related confidence intervals are computed. The tests behave like the classical sequential probability-ration test when the mean is small and like a fixed-time test when the mean is large and accurate estimation is important.

1. INTRODUCTION

An important example of hypothesis testing for the purpose of decision making is the quality control of industrial production, for which the required decision frequently is acceptance or rejection of a production lot. In such cases sequential testing offers certain advantages over fixed-sample-size tests. Generally speaking, sequential tests of the same Type I and Type II error probabilities (producer's risk and consumer's risk) require on the average fewer data to reach a decision than do their fixed-sample counterparts, and this shorter (average) testing time is usually economically desirable.

There may be cases in which one would like to supplement the decision provided by a test of a hypothesis with a point or interval estimate for some parameter. An example is the testing of a prototype with the goal of deciding whether to start production or to continue development. In the case in which a decision to start production is made, it may be advisable to have reliability estimates which are accurate enough to permit subsequent comparisons of the field reliability of production items with the laboratory reliability of the prototypes. In such cases the advantages of sequential tests are not so apparent, for the early termination of a sequential test, which seems advantageous from a decision-making point of view, may result in insufficient data for accurate estimation.

The goal of this note is to show by a concrete example that sequential tests can be adapted so that some of their advantages are maintained while their concomitant disadvantages are minimized from the viewpoint of estimation.

2. A SEQUENTIAL TEST AND THE RELATED CONFIDENCE INTERVALS

Assume that the times to failure of nominally identical items form a sequence $x_1, x_2, \ldots$ of independent exponentially distributed random variables with mean time between failures (mean lifetime) equal to θ. Suppose that to determine the acceptability of these items one

*Prepared under Contract N00014-77-C-0306 (NR-042-373) for the Office of Naval Research

desires to test the hypothesis H_0: $\theta \geq \theta_0$ vs H_1: $\theta \leq \theta_1$ with some prespecified error probabilities $\alpha = P_{\theta=\theta_0}$ (reject H_0) and $\beta = P_{\theta=\theta_1}$ (accept H_0). If H_0 is rejected, the items are deemed unacceptable, and thus α denotes the producer's risk. Similarly, β is the consumer's risk. By a change of scale it may be assumed the $\theta_1 = 1$, and indeed it will be in the rest of this note. To simplify the discussion it will be convenient to assume that one item is put on test, replaced, when it fails, by a second item, etc. until the test is terminated.

Let $s_n = x_1 + \ldots + x_n$ denote the time of the n th failure. The customary fixed-time test of H_0 against H_1 rejects H_0 if and only if, for appropriately chosen m_0 and t_0, $s_{m_0} \leq t_0$. The test parameters m_0 and t_0 are chosen to give the desired values of α and β. It is sometimes more convenient to describe the test in terms of $X(t)$, the number of failures prior to time t, which is a Poisson process of intensity $\lambda = \theta^{-1}$. The rejection region is given in terms of $X(t)$ by $X(t_0) \geq m_0$. Operationally, the test is usually censored, i.e., one observes the process of failures until the m_0th failure or for t_0 units of time, whichever occurs first. Censoring has no effect on the accept-reject decision and hence no effect on the error probabilities. It does, however, mean that the actual time on test is a random variable $\leq t_0$. Nevertheless, the term "fixed-time test" will be used to describe the censored version in order to distinguish it easily from the truly sequential tests described below. It is important to note that the confidence intervals obtained from a (censored) fixed-time test are not in general the same as from a true fixed-time test unless an accept decision is reached.

Suppose now that in addition to testing H_0 one desires to give a confidence interval for the mean time between failures θ (or equivalently for the Poisson intensity λ). Suppose also that this confidence interval is of primary interest when H_0 is accepted. When H_0 is rejected, further development and testing will be necessary; so it is more important to reach a reject decision as soon as possible than to provide an extremely accurate estimate.

The following test of H_0 against H_1 is designed to perform like a sequential probability-ratio test when H_1 is true and early termination is desired and like a fixed-length test when H_0 is true and accurate estimation is of primary concern. Let $l(t) = X(t) \log(\lambda_1/\lambda_0) - (\lambda_1 - \lambda_0)t$ be the log likelihood ratio for testing the simple hypothesis $\lambda = \lambda_0 (= \theta_0^{-1})$ against $\lambda = \lambda_1 (= \theta_1^{-1} = 1)$. Define the stopping rule, $T =$ smallest value to t such that $l(t) \geq c$. Given t_1, stop testing at min (T, t_1) and reject H_0 if and only if $T \leq t_1$. It is convenient to put $a = c/\log(\lambda_1/\lambda_0)$ and $b = (\lambda_1 - \lambda_0)/\log(\lambda_1/\lambda_0)$, so that

$$T = \text{smallest value of } t \text{ such that } X(t) \geq a + bt \tag{1}$$

(see Figure 1). The Type I and Type II error probabilities are $P_{\lambda=\lambda_0}\{T \leq t_1\}$ and $P_{\lambda=1}\{T > t_1\}$. The parameters a and t_1 should be chosen to make these equal to the desired α and β.

Computation of the error probabilities for this test is more complicated than for the customary fixed-length test. A method which gives adequate approximations for practical purposes is described in the appendix. The resulting approximations to the Type II and Type I error probabilities are respectively

$$P_1\{T > t_1\} \approx P_1\{s_m > t_1\} - (\lambda''/\lambda')^{a+bt_1} \exp[t_1(\lambda'/\lambda'' - 1)] P_{\lambda'/\lambda''}\{s_m > t_1\} \tag{2}$$

and

$$P_{\lambda_0}\{T \leq t_1\} \approx \lambda_0^a [(b-\lambda_0)/(1-b) - \lambda_0 b^{-1} P_1\{T > t_1\}]. \tag{3}$$

In equation (2), λ' and λ'' are defined by

$$\lambda' t_1 = a + bt_1 \quad \text{and} \quad b = (\lambda' - \lambda'')/\log(\lambda'/\lambda''), \tag{4}$$

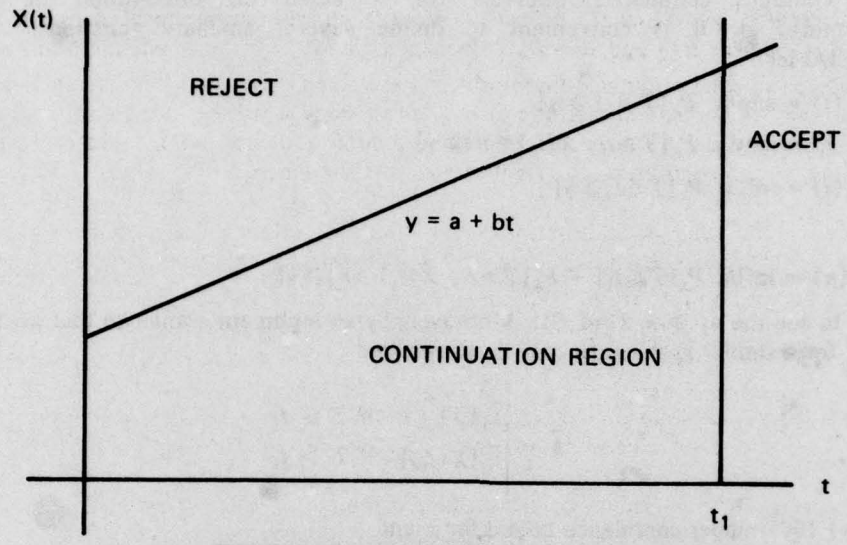

FIGURE 1. Stopping rule T and sequential test.

and m is the smallest integer $\geqslant a + bt_1$. Also, P_λ denotes probability when the true intensity of $X(t)$ is λ, so $P_\lambda\{s_m > t_1\}$ is the probability that a χ^2 random variable with 2m degrees of freedom exceeds $2\lambda t_1$. To attain given values $\alpha = P_{\lambda_0}\{T \leqslant t_1\}$ and $\beta = P_1\{T > t_1\}$ using (2) and (3), first set $P_1\{T > t_1\}$ equal to the desired value β on the right-hand side of (3) and then solve for a in terms of α and β. The value of t_1 may easily be determined from (2) by trial and error. Table 1 compares two (censored) fixed-length tests with the corresponding sequential tests. For both examples $\alpha = \beta = 0.10$. The sequential tests require on the average about 30-percent less testing time than do the fixed-length tests when $\lambda = 1$, and it is desirable to minimize testing time. They pay for this by requiring about 30-percent more maximum testing time. The maximum testing time is attained whenever H_0 is accepted, and in this case one desires to have an accurate estimate of λ. The fact that more time on test generally leads to more accurate estimation of λ tends to minimize the disadvantage of a larger maximum test duration. Nevertheless, a modified sequential test in which the maximum time on test is the same as in the fixed-length test will be described in Section 3.

TABLE 1. Comparison of (censored) fixed-length and sequential tests

	$\theta_0 = 2$ ($\lambda_0 = 1/2$)		$\theta_0 = 1.5$ ($\lambda_0 = 2/3$)	
	Fixed Length	Sequential	Fixed Length	Sequential
Test Parameters	$m_0 = 14$ $t_0 = 18.9$	$a = 2.86$, $b = 0.721$, $m = 20$ $t_1 = 23.8$, $\lambda' = 0.842$, $\lambda'' = 0.613$	$m_0 = 40$ $t_0 = 48.3$	$a = 5.1$, $b = 0.822$, $m = 58$ $t_1 = 64.3$, $\lambda' = 0.901$, $\lambda'' = 0.748$
Expected Testing Time When $\lambda = 1$	13	8.9	36	25

To compute confidence intervals for λ based on observation of $X(t)$ for $0 \leqslant t \leqslant \min(T,t_1)$, it is convenient to define several auxiliary functions. For fixed $0 < \gamma \leqslant 1/2$ let

$$\bar{\lambda}_1(t) = \sup[\lambda: P_\lambda\{T \geqslant t\} \geqslant \gamma] ,$$

$$\bar{\lambda}_2(n) = \sup[\lambda: P_\lambda\{T > t_1,\ X(t_1) \leqslant n\} \geqslant \gamma] ,$$

$$\underline{\lambda}_1(t) = \inf[\lambda: P_\lambda\{T \leqslant t\} \geqslant \gamma] ,$$

and

$$\underline{\lambda}_2(n) = \inf[\lambda: P_\lambda\{T \leqslant t_1\} + P_\lambda\{T > t_1,\ X(t_1) \geqslant n\} \geqslant \gamma] .$$

It is easy to see the $\bar{\lambda}_i \geqslant \underline{\lambda}_i$ $(i=1,2)$. Moreover, by an argument similar to that given by Siegmund [5] for a similar problem, it may be shown that

$$\bar{\lambda} = \begin{cases} \bar{\lambda}_1(T) & \text{if } T \leqslant t_1 \\ \bar{\lambda}_2[X(t_1)] & \text{if } T > t_1 \end{cases} \tag{5}$$

is a $(1 - \gamma)$ 100% upper confidence bound for λ and

$$\underline{\lambda} = \begin{cases} \underline{\lambda}_1(T) & \text{if } T \leqslant t_1 \\ \underline{\lambda}_2[X(t_1)] & \text{if } T > t_1 \end{cases} \tag{6}$$

is a $(1 - \gamma)$ 100% lower confidence bound for λ. Also $[\underline{\lambda}, \bar{\lambda}]$ is a $(1 - 2\gamma)$ 100% confidence interval.

Computation of the probabilities entering into the definitions of $\underline{\lambda}$ and $\bar{\lambda}$ involves the same problems as computation of the error probabilities, and it is discussed in detail in the appendix. For many of the cases of greatest interest, i.e., when H_0 is accepted, $\bar{\lambda}$ and $\underline{\lambda}$ are almost the same as the customary confidence limits based on the same data. For example, suppose that the sequential test defined above terminated at time t_1 with $X(t_1) = n$. Then finding an upper confidence bound for λ involves computing

$$P_\lambda\{T > t_1,\ X(t_1) \leqslant n\} = P_\lambda\{X(t_1) \leqslant n\} - P_\lambda\{T < t_1,\ X(t_1) \leqslant n\} . \tag{7}$$

If a fixed-time test were to lead to this same data, the appropriate $(1 - \gamma)$ 100% upper confidence bound for λ would be $\lambda^*[X(t_1)]$, where $\lambda^*(n) = \sup[\lambda: P_\lambda\{X(t_1) \leqslant n\} \geqslant \gamma]$. Whenever the second probability on the right-hand side of (7) is small compared to the first probability, $\bar{\lambda}_2(n)$ will be about equal to $\lambda^*(n)$, and an appropriate confidence bound based on the sequential test will be about the same as the usual confidence bound. This is presumably the case unless n is quite close to $a + bt_1$. As an example, consider the first sequential test described in Table 1, for which $a + bt_1 = 20$, and suppose that $X(t_1) = 17$. The usual 80-percent confidence interval for λ would be [0.50, 0.945], while the sequential-test-based confidence interval is $[\underline{\lambda}, \bar{\lambda}] = [0.47, 0.92]$. Although the usual interval is based on the hypothesis that a fixed-length test was being used and is not, strictly speaking, correct under the present conditions, it is probably accurate enough for practical purposes. For $X(t_1) < 17$ it will be even more accurate. If $T \leqslant t_1$, there is no reason to believe that the usual confidence interval based on the same data will approximate the interval $[\underline{\lambda}, \bar{\lambda}]$ defined in (5) and (6).

3. OTHER SEQUENTIAL TESTS

The sequential test suggested above has one important disadvantage *vis a vis* the customary fixed-length test, to wit, its maximum test duration is greater. One can argue that this is actually an advantage, because the maximum test time is attained when H_0 is accepted and

accurate estimation of λ is of primary importance. Nevertheless, it is interesting to note that one can find tests similar in spirit to the sequential test of Section 2 with no increase in the maximum testing time over the usual fixed-length tests.

Suppose that t_0 and m_0 define a fixed-length test as in Section 2 and let the stopping rule T be defined by (1) with $b = -(1-\lambda_0)/\log \lambda_0$ as before, but with a to be determined. Consider the class of sequential tests which stop testing at $\min(T,t_0)$, and for some $m_0^* < a + bt_0$ reject H_0 if and only if $T \leqslant t_0$ or $T > t_0$ and $X(t_0) \geqslant m_0^*$. The Type I and Type II error probabilities are, respectively,

$$P_{\lambda_0}\{T \leqslant t_0\} + P_{\lambda_0}\{T > t_0,\ X(t_0) \geqslant m_0^*\}$$

and

$$P_1\{T > t_0,\ X(t_0) < m_0^*\},$$

and the two free test parameters a amd m_0^* are to be chosen to make these error probabilities equal to the desired α and β. Approximate computation of these probabilities presents essentially the same problems as the determination of the confidence intervals of the preceding section and may be handled by the methods of the appendix. Unfortunately, a trial and error determination of a and m_0^* is a little more complicated than determination of a and t_1 in Section 2. Nevertheless, it can be accomplished fairly quickly by starting from the values of a for the sequential test and m_0 for the fixed length test in Section 2 and increasing both slightly to obtain trial values of a and m_0^*.

Corresponding to the first test in Table 1 ($\lambda_0 = 1/2$, $\alpha = \beta = 0.1$, and $t_0 = 18.9$), for $a = 3$ and $m_0^* = 15$ the test defined above has approximate error probabilities $\alpha = 0.099$ and $\beta = 0.109$, which are probably close enough to their nominal values for practical purposes. The expected testing time when $\lambda = 1$ is about 9.5, which is only slightly larger than that for the sequential test of Section 2 and still considerably smaller than that for the fixed-time test. Computation of confidence intervals for this test is exactly the same as for the sequential test of Section 2. It is interesting to note that whenever both this test and the fixed-time test accept H_0 (i.e., $T > t_0$ and $X(t_0) < m_0$) the confidence limits for the sequential test are shifted slightly towards smaller values of λ (larger values of θ) than the customary intervals, but the difference is probably not important for practical purposes.

Until now it has been assumed that the desire to estimate λ accurately when H_0 is true is of primary consideration and that early termination of the test under H_0 is relatively unimportant. Hence, the proposed sequential tests have been designed to behave like a sequential probability-ratio test under H_1 and like a fixed-time test under H_0. It is undoubtedly possible to obtain somewhat earlier termination for small values of λ without an appreciable loss of estimation accuracy, but whether the benefits are worth the additional complication remains for a future study to decide.

One possibility is to introduce a lower stopping boundary $-c+bt$, so that testing terminates if $X(t) \geqslant a + bt$, $X(t) \leqslant -c + bt$, or $t = t_2$ for some suitably chosen a, c, and t_2. This stopping rule is similar to the usual (truncated) sequential probability-ratio test. However, by choosing c fairly large, one does not reduce the sampling time under H_0 too much and accurate estimation is still possible. Confidence intervals may be defined similarly to those above, and it is hoped that the approximate computational methods of the appendix may be adapted to this case as well.

APPENDIX

The following theorem is the theoretical basis for (2) and the calculations required to determine $\bar{\lambda}_i$ $(i=1,2)$. It is related to Proposition 1 of Siegmund [5] (see also Woodroofe [6]). The approximation (3) is based on a heuristic argument which is given following the proof of Theorem 1. An approximation to the expected sample size for $\lambda > \lambda'$ is suggested at the end of this appendix.

THEOREM 1: Let T be defined by (1). Let m denote the least integer $\geq a + bt_1$. Assume that $a \to \infty$ and $t_1 \to \infty$ in such a way that $\Delta = m - (a + bt_1)$ remains constant and for some $\lambda' > b$

$$\lambda' t_1 = a + bt_1 \ . \tag{8}$$

Define $\lambda'' < b$ by

$$b = (\lambda' - \lambda'')/\log(\lambda'/\lambda'') \ . \tag{9}$$

Then for each $\lambda > 0$, $k = 1, 2, \ldots$, as $a \to \infty$,

$$P_\lambda\{T < t_1, X(t_1) = m-k\} \sim (\lambda''/\lambda')^{a+bt_1} \exp[\lambda t_1(\lambda'/\lambda'' - 1)] P_{\lambda\lambda'/\lambda''}\{X(t_1) = m-k\} \ .$$

PROOF: By the definition of conditional probability,

$$P_\lambda\{T < t_1, X(t_1) = m-k\} = P_\lambda\{T < t_1 | X(t_1) = m-k\} P_\lambda\{X(t_1) = m-k\} \ . \tag{10}$$

It follows from Lemmas 1 and 3 below that the conditional probability in (10) converges to $(\lambda''/\lambda')^{k-\Delta}$ as $a \to \infty$. The theorem follows by substitution of the values of $P_\lambda\{X(t_1) = m-k\}$.

LEMMA 1: $P_\lambda\{T < t_1 | X(t_1) = m-k\}$ does not depend on λ, and under the conditions of Theorem 1 converges to $P_{\lambda'}\{X(t) \leq -k+\Delta+bt \text{ for some } t > 0\}$ as $a \to \infty$.

PROOF: That $P_\lambda\{T < t_1 | X(t_1) = m-k\}$ does not depend on λ is an immediate consequence of the sufficiency of $X(t_1)$. Hence, it suffices to give the proof for some particular value of λ. Also, for $0 < r < t_1$

$$\begin{aligned} &P_\lambda\{X(t) \geq a+bt_1 \text{ for some } t_1 - r \leq t < t_1 | X(t_1) = m-k\} \leq P_\lambda\{T < t_1 | X(t_1) = m-k\} \\ &\leq P_\lambda\{T_1 < t_1 - r | X(t_1) = m-k\} + P_\lambda\{X(t_1) \geq a+bt \text{ for some } t_1 - r \leq t < t_1 | X(t_1) = m-k\}. \end{aligned} \tag{11}$$

By symmetry the first probability in (11) equals

$$\begin{aligned} &P_\lambda\{X(t_1) - X(t) \leq -k+\Delta+b(t_1-t) \text{ for some } t_1 - r \leq t < t_1 | X(t_1) = m-k\} \\ &= P_\lambda\{X(t) \leq -k+\Delta+bt \text{ for some } 0 < t \leq r | X(t_1) = m-k\} \ . \end{aligned}$$

It is easy to see by direct calculation that, conditional on $X(t_1) = m - k \sim \lambda' t_1$, the process $X(t)$, $0 \leq t \leq r$, converges to a Poisson process of intensity λ'. Hence, the preceding conditional probability converges to

$$P_{\lambda'}\{X(t) \leq -k+\Delta+bt \text{ for some } 0 < t \leq r\} \ .$$

Since r is arbitrary, to complete the proof it suffices, by (11), to show that

$$P_\lambda\{T < t_1 - r | X(t_1) = m-k\} \leq \epsilon(r) \ , \tag{12}$$

where $\epsilon(r)$ does not depend on a (or t_1) and converges to 0 as $r \to \infty$. First observe that

$$P_\lambda\{X(t_1)=m-k\} = \exp[(\lambda'-\lambda)t_1](\lambda/\lambda')^{m-k}P_{\lambda'}\{X(t_1)=m-k\},$$

and by the local limit theorem and (8)

$$P_{\lambda'}\{X(t_1)=m-k\} \sim (2\pi m)^{-1/2}.$$

Hence, to prove (12) it suffices to show for some $\lambda > 0$

$$P_\lambda\{T \leqslant t_1 - r\} \leqslant \epsilon(r)\exp[(\lambda'-\lambda)t_1](\lambda/\lambda')^{m-k}m^{-1/2}. \tag{13}$$

The proof of (13) is simplified notationally if we assume that $m = a + bt_1$ and take r to be such that $br = j$ is an integer. The general case requires only slight changes. Then

$$\begin{aligned}
P_\lambda\{T \leqslant t_1 - r\} &= P_\lambda\{X(t) \geqslant a+bt \text{ for some } t \leqslant t_1 - r\} \\
&= P_\lambda\{n \geqslant a+b\,s_n \text{ for some } n \leqslant m-j\} \\
&\leqslant \sum_{a\leqslant n\leqslant m-j} P_\lambda\{s_n \leqslant b^{-1}(n-a)\} = \sum_{a\leqslant n\leqslant m-j} P_\lambda\{X[b^{-1}(n-a)] \geqslant n\} \\
&= \sum_{a\leqslant n\leqslant m-j}\ \sum_{i\geqslant n} \exp[(\lambda'-\lambda)b^{-1}(n-a)](\lambda/\lambda')^i P_{\lambda'}\{X[b^{-1}(n-a)] = i\}.
\end{aligned}$$

It may be shown for all $t \leqslant t_1$ and $i \geqslant a -$ by standard large deviation theory for $t \leqslant \epsilon t_1$ (Bahadur and Rao [1]) and by a local limit theorem for $\epsilon t_1 \leqslant t \leqslant t_1 -$ that $P_{\lambda'}\{X(t) = i\} \leqslant Cm^{-1/2}$. Hence, for all $\lambda < \lambda'$

$$\begin{aligned}
P_\lambda\{T \leqslant t_1 - r\} &\leqslant C(1-\lambda/\lambda')^{-1}m^{-1/2}\exp[(\lambda-\lambda')b^{-1}a] \sum_{a\leqslant n\leqslant m-j} \{\exp[(\lambda'-\lambda)b^{-1}](\lambda/\lambda')\}^n \\
&= C(1-\lambda/\lambda')^{-1}m^{-1/2}(\lambda/\lambda')^{m-j} \\
&\cdot \exp[(\lambda-\lambda')(-t_1+b^{-1}j)] \sum_{a\leqslant n\leqslant m-j} \{\exp[(\lambda'-\lambda)b^{-1}](\lambda/\lambda')\}^{n-m+j}.
\end{aligned}$$

By (9), for $\lambda'' < \lambda < \lambda'$, $\exp[(\lambda'-\lambda)b^{-1}](\lambda/\lambda') > 1$, and hence the indicated sum above can be extended as a convergent series to $n = -\infty$. Comparing the resulting inequality with (13) and recalling the $j = br$, we can complete the proof.

The following lemmas are known, but for lack of a convenient reference and for completeness their proofs are sketched.

LEMMA 2: Let μ_1, μ_2 be positive numbers and τ a stopping rule for $X(t)$, $0 \leqslant t < \infty$. Then for arbitrary $t > 0$

$$P_{\mu_1}\{\tau \leqslant t\} = \int_{\{\tau\leqslant t\}} \exp[(\mu_2-\mu_1)\tau](\mu_1/\mu_2)^{X(\tau)}\,dP_{\mu_2}. \tag{14}$$

PROOF: Let $P_\lambda^{(t)}$ denote the restriction of P_λ to the space of $X(s)$, $0 \leqslant s \leqslant t$, and let $Z(t) = \exp[(\mu_2-\mu_1)t](\mu_1/\mu_2)^{X(t)}$ be the likelihood ratio of $P_{\mu_1}^{(t)}$ relative to $P_{\mu_2}^{(t)}$. Then

$$P_{\mu_1}\{\tau > t\} = \int_{\{\tau>t\}} Z(t)\,dP_{\mu_2}.$$

Since $Z(t)$, $0 \leqslant t < \infty$, is a P_{μ_2} – martingale,

$$1 = E_{\mu_2}\{Z[\min(\tau,t)]\} = \int_{\{\tau<t\}} Z(\tau)\,dP_{\mu_2} + \int_{\{\tau>t\}} Z(t)\,dP_{\mu_2},$$

and (14) follows by subtraction:

LEMMA 3: For each $x > 0$ and $\lambda' > b$

$$P_{\lambda'}\{X(t) \leqslant -x+bt \text{ for some } t>0\} = (\lambda''/\lambda')^x ,$$

where $\lambda'' < b$ is defined by (9).

PROOF: Let $\tau = \inf[t: X(t) \leqslant -x+bt]$, and observe that with probability 1

$$X(\tau) = -x + b\tau \text{ on } (\tau<\infty) . \tag{15}$$

Letting $t \to \infty$ in (14) and appealing to (9) and (15) we obtain

$$\begin{aligned} P_{\lambda'}\{\tau<\infty\} &= \int_{\{\tau<\infty\}} \exp[(\lambda''-\lambda')\tau](\lambda'/\lambda'')^{X(\tau)}\, dP_{\lambda''} \\ &= (\lambda''/\lambda')^x\, P_{\lambda''}\{\tau<\infty\} . \end{aligned}$$

Since $\lambda'' < b$, $P_{\lambda''}\{\tau<\infty\} = 1$, which completes the proof.

COROLLARY 1: For $\lambda > \lambda'$ and $k = 1, 2, \ldots$, under the conditions of Theorem 1

$$\begin{aligned} P_\lambda\{T\leqslant t_1,\ X(t_1)\leqslant m-k\} &\sim (\lambda''/\lambda')^{a+bt_1} \\ &\cdot \exp[\lambda t_1(\lambda'/\lambda''-1)]P_{\lambda\lambda'/\lambda''}\{X(t_1)\leqslant m-k\} . \end{aligned} \tag{16}$$

PROOF: Obviously for all $j > 0$

$$\begin{aligned} P_\lambda\{T\leqslant t_1,\ m-k-j<X(t_1)\leqslant m-k\} &\leqslant P_\lambda\{T\leqslant t_1,\ X(t_1)\leqslant m-k\} \\ &\leqslant P_\lambda\{T\leqslant t_1,\ m-k-j<X(t_1)\leqslant m-k\} + P_\lambda\{X(t_1)\leqslant m-k-j\} . \end{aligned} \tag{17}$$

For fixed j the probability on the left and the first probability on the right of (17) may be evaluated asymptotically by appealing to Theorem 1. The second term on the right hand side of (17) may be shown to be of smaller order of magnitude for large j by standard large-deviation arguments. See Bahadur and Rao [1] or Siegmund [4] for the details of similar arguments.

REMARKS: (i) Corollary 1 may be used to evaluate approximately the probability in (2) and the probabilities appearing in the definitions of $\bar{\lambda}_i$ by virtue of the relation

$$P_\lambda\{T>t_1,\ X(t_1)\leqslant n\} = P_\lambda\{X(t_1)\leqslant n\} - P_\lambda\{T\leqslant t_1,\ X(t_1)\leqslant n\} .$$

The approximation can be expected to yield accurate results only when n is close to $a + bt_1$, i.e., $n = m - k$ for a fairly small k, and $\lambda > \lambda'$. But these are essentially the only values of interest, as was observed in Section 2.

(ii) A slightly more sophisticated argument similar to that given by Siegmund [5] shows that (16) holds for all $\lambda > \lambda''$, $\lambda \neq \lambda'$.

(iii) It is tempting to evaluate the probability appearing in the definition of $\underline{\lambda}_2(n)$ by rewriting it as

$$P_\lambda\{X(t_1)\geqslant n\} + P_\lambda\{T\leqslant t_1,\ X(t_1)\leqslant n\}$$

and then applying Corollary 1. However, this evaluation is of interest for small λ when Corollary 1 is not applicable.

Now suppose that $\lambda_0 < \lambda''$, and let $\lambda_1 > \lambda'$ be defined by

$$(\lambda_1-\lambda_0) = b\log(\lambda_1/\lambda_0) . \tag{18}$$

These values λ_0 and λ_1 may be the values specified by the test of the hypothesis of Section 2 but need not be. The approximation (3) and an approximate evaluation of the probabilities appearing in the definitions of $\underline{\lambda}_i$ $(i=1,2)$ may be obtained if we write

$$P_{\lambda_0}\{T\leqslant t_1\} = P_{\lambda_0}\{T<\infty\} - P_{\lambda_0}\{t_1<T<\infty\} \tag{19}$$

and give separate approximations for the two terms on the right hand side of (19). Letting $t\to\infty$ in Lemma 2 (note the $P_{\lambda_1}\{T<\infty\}=1$) and appealing to (18) we obtain the representations

$$P_{\lambda_0}\{T<\infty\} = (\lambda_0/\lambda_1)^a\, E_{\lambda_1}[(\lambda_0/\lambda_1)^{X(T)-a-bT}]$$

and

$$P_{\lambda_0}\{t_1<T<\infty\} = (\lambda_0/\lambda_1)^a \int_{\{T>t_1\}} E_{\lambda_1}[(\lambda_0/\lambda_1)^{X(T)-a-bT}\,|\,X(t_1)]\,dP_{\lambda_1}\,. \tag{20}$$

The limiting value of $E_{\lambda_1}[(\lambda_0/\lambda_1)^{X(T)-a-bt}]$ as $a\to\infty$ is given in Lemma 4 below leading to the asymptotic relation

$$P_{\lambda_0}\{T<\infty\} \sim [(b-\lambda_0)/(\lambda_1-b)](\lambda_0/\lambda_1)^a \text{ as } a\to\infty\,. \tag{21}$$

The integral in (20) may be rewritten as

$$\sum_{K=1}^{m} P_{\lambda_1}\{T_a>t_1,\ X(t_1)=m-k\}\, E_{\lambda_1}[(\lambda_0/\lambda_1)^{X(T_{k-\Delta})-k+\Delta-bT_{k-\Delta}}]\,, \tag{22}$$

where $T_x=\inf[t:\ t>0,\ X(t)\geqslant x+bt]$ and $\Delta=m-(a+bt_1)$ as before. To obtain a simple approximation to (22), one might replace the expectations there by their limits as $k\to\infty$, or one might replace all these expectations by that for T_0, which is also given in Lemma 4. The latter alternative seems more attractive for two reasons: (a) the values of $P_{\lambda_1}\{T>t_1,\ X(t_1)=m-k\}$ are larger for smaller values of k, and (b) using $k=0$ rather than k infinitely large we obtain a smaller approximation to (22) and hence a larger (more conservative) approximation to $P_{\lambda_0}\{T\leqslant t_1\}$. Replacing the $T_{k-\Delta}$ by T_0 in (22) leads, by Lemma 4 below, to the approximation

$$P_{\lambda_0}\{t_1<T<\infty\} \approx \lambda_0 b^{-1} P_{\lambda_1}\{T>t_1\}(\lambda_0/\lambda_1)^a\,.$$

Substituting this result and (21) into (19), we obtain

$$P_{\lambda_0}\{T\leqslant t_1\} \approx (\lambda_0/\lambda_1)^a\,[(b-\lambda_0)/(\lambda_1-b)-\lambda_0 b^{-1}P_{\lambda_1}\{T>t_1\}]\,,$$

which is (3) in the special case $\lambda_1=1$.

LEMMA 4: For λ_0 and λ_1 defined by (18)

$$\lim_{a\to\infty} E_{\lambda_1}\,[(\lambda_0/\lambda_1)^{X(T_a)-a-bT_a}] = (b-\lambda_0)/(\lambda_1-b)$$

and

$$E_{\lambda_1}\,[(\lambda_0/\lambda_1)^{X(T_0)-bT_0}] = \lambda_0/b\,.$$

PROOF: Let $\tau_a=\inf[n:\ n-bs_n\geqslant a]$, so $X(T_a)=\tau_a$ and $T_a=s_{\tau_a}$. Also, let $\tau_-=\inf[n:\ n-bs_n<0]$. By Lemma 2 and random walk theory (Feller [2], Chapter XII),

$$E_{\lambda_1}[(\lambda_0/\lambda_1)^{X(T_0)-bt_0}] = P_{\lambda_0}\{T_0<\infty\} = 1-P_{\lambda_0}\{\tau_0=\infty\}$$

$$= 1-1/E_{\lambda_0}[\tau_-] = 1-(1-b\lambda_0^{-1})/E_{\lambda_0}[\tau_- - bs_{\tau_-}] = 1+(1-b\lambda_0^{-1})/b\lambda_0^{-1} = \lambda_0/b\,.$$

By considering the renewal process defined by $\tau_0 - bs_{\tau_0}$, the renewal theorem gives the P_{λ_1} limiting distribution of $\tau_a - bs_{\tau_a} = X(T_a) - bT_a$ as $a \to \infty$ (Feller [2], Chapter XI). Combined with (18) and random-walk theory (Feller [2], Chapter XII), this leads to

$$\lim_{a\to\infty} E_{\lambda_1}[(\lambda_0/\lambda_1)^{X(T_a)-a-bT_a}] = \lim_{a\to\infty} E_{\lambda_1} \exp[-b^{-1}(\lambda_1-\lambda_0)(\tau_a - bs_{\tau_a})]$$

$$= \{E_{\lambda_1}[\tau_0 - bs_{\tau_0}]\}^{-1} \int_0^\infty \exp[-b^{-1}(\lambda_1-\lambda_0)x] P_{\lambda_1}\{\tau_0 - bs_{\tau_0} > x\} dx$$

$$= [(1-b/\lambda_1)E_{\lambda_1}\tau_0]^{-1}\, b(\lambda_1-\lambda_0)^{-1}\, P_{\lambda_0}\{\tau_0 = \infty\}$$

$$= b(\lambda_1-\lambda_0)^{-1}(1-b/\lambda_1)^{-1}(E_{\lambda_0}\tau_-)^{-1}\, P_{\lambda_1}\{\tau_- = \infty\}$$

$$= b(\lambda_1-\lambda_0)^{-1}(1-b/\lambda_1)^{-1}(1-\lambda_0/b)(1-\lambda_0/\lambda_1)$$

$$= (b-\lambda_0)/(\lambda_1-b)\ .$$

There remains the task of computing approximations to $E_\lambda[\min(T,t_1)]$, especially for $\lambda \geqslant 1$. With the help of Theorem 1, this becomes an easy matter. It is easy to see that for $\lambda > b$

$$E_\lambda[\min(T,t_1)] = E_\lambda[T] - \int_{\{T>t_1\}} E_\lambda[T - t_1 | X(t_1)]\, dP_\lambda\ . \tag{23}$$

The usual Wald approximation to $E_\lambda[T]$ obtained by using Wald's identity and ignoring the excess $X(T) - (a+bT)$ is

$$E_\lambda[T] \approx a/(\lambda - b)\ . \tag{24}$$

The same argument gives the approximation

$$E_\lambda[T - t_1 | X(t_1) = m - k] \approx (k-\Delta)/(\lambda - b)\ . \tag{25}$$

Hence the second term on the right-hand side of (23) may be rewritten

$$\sum_{k=1}^{m} E_\lambda[T - t_1 | X(t_1) = m-k][P_\lambda\{X(t_1) = m-k\} - P_\lambda\{T \leqslant t_1,\ X(t_1) = m-k\}]$$

and evaluated approximately with the help of (25) and Theorem 1 to obtain

$$\int_{\{T>t_1\}} E_\lambda[T - t_1 | X(t_1)]\, dP_\lambda \approx (\lambda - b)^{-1}[(a+bt_1)P_\lambda\{T > t_1\}$$

$$-\lambda t_1 P_\lambda\{s_{m-1} > t_1\} + (\lambda''/\lambda')^{m-1}\, \lambda t_1 \exp\{\lambda t_1(\lambda'/\lambda'' - 1)\} P_{\lambda\lambda'/\lambda''}\{s_{m-1} > t_1\}]\ . \tag{26}$$

Substituting (24) and (26) into (23) gives an approximation to $E_\lambda[\min(T,t_1)]$. It should be fairly accurate, at least whenever $\lambda > \lambda'$, so $P_\lambda\{T > t_1\}$ is small and the second term on the right hand side of (23) is small compared with the first. It is possible to approximate the expected excess over the boundary $E_\lambda[X(T) - a - bT]$, and hence give a slightly more accurate approximation. However, the increase in accuracy is rather modest in contrast to the problem of approximating error probabilities, where the excess over the boundary is more important (see Siegmund [3]).

REFERENCES

[1] Bahadur, R.R., and Rao, R. Ranga, "On Deviations of the Sample Mean," *Annals of Mathematical Statistics, 31*, 1015-1027 (1960).

[2] Feller, W., *An Introduction to Probability Theory and Its Applications,* Vol. II (Wiley, New York, 1966).

[3] Siegmund, D., "Error Probabilities and Average Sample Number of the Sequential Probability Ratio Test," Journal of the Royal Statistical Society, Ser. B *37*, 394-401 (1975).

[4] Siegmund, D., (1975b). "Large deviation probabilities in the strong law of large numbers," *Zeitschrift für Wahrscheinlichkeitstheorie und Verwendte Gebiete 31,* 107-113 (1975).

[5] Siegmund, D. "Estimation Following Sequential Tests" *Biometrika 65*, 341-349 (1978).

[6] Woodroofe, M., "A Renewal Theorem for Curved Boundaries and Moments of First Passage Times," *Annals of Probability, 4*, 67-80 (1976).

INTERVAL ESTIMATION OF A GLOBAL OPTIMUM FOR LARGE COMBINATORIAL PROBLEMS

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ABSTRACT

Consider an "intractable" optimization problem for which no efficient solution technique exists. Given a systematic procedure for generating independent heuristic solutions, we seek to obtain interval estimates for the globally optimal solution using statistical inference. In previous work, accurate point estimates have been derived. Determining interval estimates, however, is a considerably more difficult task. In this paper, we develop straightforward procedures which compute confidence intervals efficiently in order to evaluate heuristic solutions and assess deviations from optimality. The strategy presented is applicable to a host of combi atorial optimization problems. The assumptions of our model, along with computational experience, are discussed.

INTRODUCTION

Many problems in the area of combinatorial optimization are known to be NP-complete. Any current algorithm for obtaining the exact solution to a problem in this class is an exponential time algorithm. That is, in the worst case, running time increases exponentially as a function of an input parameter, such as the number of nodes. Moreover, evidence suggests that polynomial time algorithms do not exist for any NP-complete problem (see Karp [12] for more details).

With this in mind, simple and efficient heuristic techniques which yield near-optimal solutions consistently become essential. At the same time, tools need to be developed and sharpened in order to measure the accuracy of these heuristics. Much recent research has been directed at generating performance guarantees for heuristic algorithms which specify a priori upper bounds (a priori in the sense that the bounds are independent of the data) on percentage deviations from optimality. Garey and Johnson [5] and Sahni [23] survey combinatorial approximation algorithms which are guaranteed to find solutions that are "close" to optimal. In earlier work, Held and Karp [10,11] had shown how to derive sharp data-dependent bounds using duality theory and Lagrangean relaxation.

In this paper, we approach the problem of analyzing heuristic solutions from a different perspective. Although we focus on the infamous traveling salesman problem (TSP), the approach applies to other NP-complete problems as well. The methods we discuss attempt to bring statistical inference to bear in order to provide data-dependent tight lower (upper) bounds for combinatorial minimization (maximization) problems. These bounds have probabilities associated with them but are easier to calculate than bounds obtained via Lagrangean relaxation.

In particular, given a systematic procedure for generating independent heuristic solutions, we seek to obtain interval estimates for the globally optimal solution using statistical inference. In recent work by Golden [6,7], Dannenbring [3], and Klein [13] accurate point estimates have been derived. Determining interval estimates, however, is a considerably more difficult task. The key objective of this paper is to develop straightforward procedures which compute confidence intervals efficiently in order to evaluate heuristic solutions and assess deviations from optimality.

Define a network $G = [N, A, C]$ with N the set of nodes, A the set of arcs, and $C = [c_{ij}]$ the matrix of costs. The TSP requires that we find the Hamiltonian cycle of G of minimal total cost. The interval estimation procedure developed will be applied to the TSP and will use the 2-opt heuristic algorithm discussed by Lin [15] for generating solutions.

Given a randomly constructed starting tour, the 2-opt algorithm checks each pair of arcs to see if it can be replaced by another pair of arcs not yet in the tour so as to produce a new tour of reduced cost. The procedure continues until no further replacements can be performed. The random starting tours are easily generated with a computer program from Nijenhuis and Wilf [20] and the entire process can be repeated any number of times.

STATISTICAL BACKGROUND

Suppose we take S independent samples, each of size m, from a parent population which is bounded from below by a. If x_i is the smallest value in sample i, then let

$$v = \min \{x_i : 1 \leqslant i \leqslant S\}. \tag{1}$$

Fisher and Tippett [4] proved that as m gets large the distribution of x_i approaches a Weibull distribution with a as the location parameter (see Gumbel [8] for more details). The cumulative Weibull distribution is given by

$$F_x(x_0) = \text{Prob}\{x \leqslant x_0\} = 1 - \exp\left\{-\left[\frac{x_0 - a}{b}\right]^c\right\} \tag{2}$$

$$\text{for } x_0 \geqslant a \geqslant 0,\ b \geqslant 0,\ c \geqslant 0,$$

where a is the location parameter, b is the scale parameter, and c is the shape parameter.

McRoberts [18] and Golden [6,7] appeal to this fundamental result in statistical extreme-value distribution theory in order to derive point estimates for the global optimum. Specifically, in [7] we argue that if G has n nodes, then the parent population consists of $(n-1)!/2$ tours with total costs bounded from below by a, the length of the optimal tour. Each heuristic solution x_i $(i = 1, 2, \ldots, S)$ is a local minimum relative to the "swapping" operation from a large number m of possible tours. Furthermore, these heuristic solutions are independent since the initial tours are randomly chosen.

Good quick estimates for the three parameters can be determined from a least-squares/goodness-of-fit analysis, as discussed by Golden [7]. One can also solve the maximum-likelihood equations using one of the methods recommended by Zanakis [26]. We remark briefly that Dannenbring [3] and Klein [13] have also approached the point estimation problem but from different viewpoints. Dannenbring studies an estimate derived by Robson and Whitlock [22] which applies to any distribution with a left truncation point. Klein's

approach involves using a three-parameter power-function distribution (a special case of the beta distribution).

In order to verify the argument that the heuristic solutions are Weibull distributed, we computed Kolmogorov-Smirnov statistics (assuming independence) in our point estimation experiments [7]. In all cases, these statistics fell far below the critical value at the 0.05 level of significance, indicating that there is no reason to reject the Weibull hypothesis. In addition, we observed that the point estimate was always within 2.8% of the true optimal solution.

Next, we sought to confirm the intuitive reasoning that heuristic solutions are independent. One set of fifty 2-opt solutions to a 100-node TSP was randomly selected, and the theory of runs was applied to this data. We describe two statistical tests. Additional tests were performed; all had the same outcome.

The first test of randomness is based on the *total* number of runs. This test dichotomizes the sequence of 50 observations by classifying each observation as either above or below the sample median. The number of runs above and below the median is listed in Table 1.

TABLE 1. *Test of Randomness*

Run Length	Number of Runs Above the Median	Number of Runs Below the Median
1	9	5
2	0	5
3	3	2
4	0	1
5	0	0
6	0	0
7	1	0
Total	13	13

Thus, the total number of runs is 26. Swed and Eisenhart's tables [25] show that the limiting values for the total number of runs above and below the median are 16 at the 0.005 level of significance and 19 at the 0.05 level of significance. Hence, there is no indication that the total number of runs above and below the median is smaller than would occur in a random sequence.

The next statistical test of randomness is based on the *length* of the longest run on either side of the median. Inspection reveals that the length of the longest run is 7. Now, Mosteller's table [19] gives the limiting values at the 0.05 and 0.01 significance levels as 10 and 11, respectively. Therefore, the length of the longest run does not give any reason to reject the hypothesis of randomness.

In summary, the run tests of randomness demonstrate that the 2-opt solutions are independent. The Kolmogorov-Smirov test convinces us that the 2-opt solutions are distributed according to a Weibull distribution. The effect of these statistical tests is to give increased credibility to the validity of point and interval estimation in this setting.

INTERVAL ESTIMATION OF A GLOBAL OPTIMUM

After point estimation, the next step is to develop a procedure for determining a confidence interval for a, the optimal TSP solution. This task is considerably more difficult than obtaining a point estimate but potentially far more rewarding. If we could say, for example, that with probability 0.99 the optimal solution is contained in the interval [975,1000] for a particular problem, then we would have an invaluable tool for evaluating heuristic solutions and measuring deviations from optimality. In this section, we discuss several candidate interval-estimation techniques. Only one of these performs successfully. We present explanations for the failure or success of each technique. In addition, various computational experiments are discussed.

Clough [2], using Monte Carlo methods, applied extreme-value theory to the problem of estimating the global optimum of a function of several variables subject to a set of constraints; he developed confidence statements as well. Unfortunately, his results pertain only to the very restricted instance of the Weibull distribution where $c = 1$ (an exponential distribution). If we can fit the heuristic solutions to an exponential distribution then, upon manipulating equation (37) in Clough [2], it can be shown that

$$\text{(3)} \qquad \text{Prob}\left\{\left(\frac{S+h}{S-1}\right)v - \left(\frac{1+h}{S-1}\right)\left(\frac{1}{S}\right)\sum_{i=1}^{S} x_i \leqslant a \leqslant v\right\} = 1 - \left(\frac{S-1}{S+h}\right)^{S-1}$$

for arbitrary values of h.

Not surprisingly, this confidence interval almost always fails to contain the optimal solution. This follows since, as we will see later, the 2-opt solutions tend to have Weibull distributions with shape parameters of approximately 2. Setting $c = 1$ beforehand invalidates the Fisher-Tippett result on which this model relies.

Assuming that the distribution of x_i is, approximately, a three-parameter Weibull distribution, Mann et al. in section 5.2.3(d) of their excellent book [17] demonstrate an interval estimation procedure which depends on a beta distribution approximation. In experiments on eight TSP's, only five of the eight resulting 90% confidence intervals included the optimum. A possible explanation for these lackluster statistics is the fact that *two* approximations (the Weibull approximation and then the beta approximation) are needed in order to construct confidence intervals. The approximation errors are very likely cumulative.

Let $x_{(i)}$ be the i^{th} order statistic (then $v = x_{(1)}$). Robson and Whitlock [22] introduced an exact $100(1-\alpha)\%$ confidence statement on a left truncation point for a uniform distribution and argued that this gives an approximate confidence interval for other density functions as well. This approximate confidence interval is given by

$$\text{(4)} \qquad \text{Prob}\left\{v - \left(\frac{1-\alpha}{\alpha}\right)(x_{(2)} - v) \leqslant a \leqslant v\right\} \approx 1 - \alpha.$$

If we set α equal to 0.5, the lower confidence limit becomes the point estimate tested by Dannenbring.

We generated fifty 2-opt solutions to the five 100-node benchmark problems first presented by Krolak et al. [14]. In addition, the 50 solutions found by Padberg and Hong [21] to the 318-node problem of Lin and Kernighan [16] using a variant of the 2-opt are also

analyzed. Table 2 reveals that this procedure also has shortcomings. Furthermore, note that if the confidence coefficient is increased from 0.95 to 0.99 then the confidence interval [15961.25, 22382.39] for problem 28 becomes uninformative.

TABLE 2. *Computational Results—Interval Estimation Using the Robson-Whitlock Approach*

Problem Identification	v	$x_{(2)}$	95% Confidence Interval	Optimal Solution	Does Interval Include Optimum?
24 [14]	21518.99	21545.79	[21009.79, 21518.99]	21282.	Yes
25 [14]	22816.55	22881.52	[21582.12, 22816.55]	22148.	Yes
26 [14]	20971.54	20975.48	[20896.68, 20971.54]	20749.	No
27 [14]	21807.23	21827.43	[21423.43, 21807.23]	21294.	No
28 [14]	22382.39	22447.25	[21150.05, 22382.39]	22068.	Yes
318 nodes [16]	41415.00	41460.00	[40560.00, 41415.00]	41349.	Yes

This procedure has a number of deficiencies. Primarily, the confidence statement is distribution-free and thus fails to exploit the known characteristics of the Weibull distribution. In particular, since the Weibull distribution is noticeably nonuniform, the approximate confidence interval is probably not appropriate in this application.

Now we present an interval estimation procedure which seems to perform quite satisfactorily. From equation (2) we see that

$$F_{x_i}(a+b) = 1 - e^{-1} \approx 0.63, \tag{5}$$

which enables us to write

$$\begin{aligned}\text{Prob}\{v < a+b\} &= 1 - \text{Prob}\{v > a+b\} \\ &= 1 - [1 - F_{x_1}(a+b)][1 - F_{x_2}(a+b)] \cdots [1 - F_{x_S}(a+b)] \\ &= 1 - e^{-S}, \text{ or}\end{aligned}$$

$$\text{Prob}\{v - b \leqslant a \leqslant v\} = 1 - e^{-S}. \tag{6}$$

Thus, a $100(1 - e^{-S})\%$ confidence interval for a is $[v - b, v]$ when b is fixed. Since b is seldom known, we now present a procedure for constructing an approximate confidence interval for the location parameter a.

STEP 1: Compute $x_1, x_2, \ldots, x_S$.

STEP 2: Rearrange the observations from smallest to largest to obtain $v = x_{(1)}, x_{(2)}, \ldots, x_{(S)}$, and find the median x_M.

STEP 3: Determine good initial parameter estimates from the expressions

$$\tilde{a} = v - (x_{(2)} - v), \tag{7}$$

$$\tilde{b} = x_{([0.63S]+1)} - \tilde{a}, \tag{8}$$

where [z] is the largest integer less than z, and

$$\check{c} = \frac{\ln[-\ln(0.5)]}{\ln(x_M - \tilde{a}) - \ln\tilde{b}}. \tag{9}$$

STEP 4: Solve the maximum-likelihood equations for "improved" parameter estimates $\tilde{a}$, $\tilde{b}$, $\check{c}$.

STEP 5: An approximate confidence interval is given by $[y - \tilde{b}, v]$.

Some comments regarding this algorithm, especially Step 3, are in order. Equations (7) and (8) follow from equations (4) and (5), respectively. The Weibull cumulative-distribution function yields (upon taking logarithms twice) the following equation

$$\ln\{-\ln[1 - F(x_0)]\} = c\ln(x_0 - a) - c\ln b. \tag{10}$$

If we replace a, b, and c with their estimates and let $x_0 = x_M$, equation (9) results. Zanakis [26] in his comparison of procedures for deriving maximum-likelihood estimates for the three-parameter Weibull distribution suggests that good initial parameter estimates may reduce the computation time and increase the accuracy of an algorithm; in addition, he finds the Harter-Moore gradient-search [9] technique to be the most accurate code for performing Step 4, in general. We use this code in our computational experiments. The data analyzed in Table 2 is analyzed again in Table 3.

TABLE 3. *Computational Results—$100(1-e^{-S})$% Confidence Intervals**

Problem Identification	v	$\tilde{a}$	$\tilde{b}$	$\check{c}$	$[v-\tilde{b}, v]$	Optimal Solution	Does Interval Include Optimum?
24 [14]	21518.99	21454.77	926.36	1.76	[20592.63, 21518.99]	21282.	Yes
25 [14]	22816.55	22754.95	856.92	1.92	[21959.63, 22816.55]	22148.	Yes
26 [14]	20971.54	20763.87	1260.75	2.31	[19710.79, 20971.54]	20749.	Yes
27 [14]	21807.23	21779.19	837.82	1.51	[20969.41, 21807.23]	21294.	Yes
28 [14]	22382.39	22190.20	1415.64	2.62	[20966.75, 22382.39]	22068.	Yes
318 nodes [16]	41415.00	41365.38	609.58	1.96	[40805.42, 41415.00]	41349.	Yes

*For these problems, $S = 50$ and $e^{-S} = 1.92874 \times 10^{-22}$. This implies that $1 - e^{-S}$ is practically one.

Notice that the width of the confidence interval ($\tilde{b}$) is never more than about 6.5% of the optimal solution and that the optimal solution is contained in the interval in every case. We expect this to be true almost always since $1 - e^{-S}$ approaches 1 very quickly as S increases. It is not surprising that this technique works well when one considers that the Weibull assumption is taken advantage of, only one estimated parameter enters into the confidence statement explicitly, and the confidence coefficient is extremely close to unity. The simplicity of the statement is an additional advantage.

Since there are so few TSP's of over 100 nodes where the optimal solution is known, we take advantage of the work of Beardwood et al. [1] who derived an asymptotic expected-length formula for an optimal traveling salesman tour when the nodes are distributed randomly and uniformly over a rectangular area of R units. The expected length of the optimal TSP tour, $L(n,R)$, is given by

(11) $$L(n,R) = K\sqrt{n}\sqrt{R}.$$

In extensive computational experiments, Stein [24] obtains the following empirical bounds on the constant K:

(12) $$0.765 \leqslant K \leqslant 0.765 + \frac{4}{n}.$$

In Table I of Golden [6], networks of from 70 to 130 nodes were generated uniformly in a square of area 10,000, heuristic solutions to the TSP were calculated, and Weibull point estimates were computed. Confidence intervals for the location parameters in these problems are developed in Table 4 where $0.765\sqrt{n}\sqrt{R}$ is taken as the presumed optimal solution.

TABLE 4. *Computational Results—$100(1-e^{-S})$% Confidence Intervals**

Problem Identification	v	$[v-\tilde{b},v]$	Optimal Solution ($76.5\sqrt{n}$)	Does Interval Include Optimum?
70 nodes [6]	659.39	[621.61, 659.39]	640.04	Yes
80 nodes [6]	700.46	[662.69, 700.46]	684.24	Yes
90 nodes [6]	747.24	[711.50, 747.24]	725.74	Yes
100 nodes [6]	785.73	[668.31, 785.73]	765.00	Yes
110 nodes [6]	832.44	[720.79, 832.44]	802.34	Yes
120 nodes [6]	915.36	[833.06, 915.36]	838.02	Yes
130 nodes [6]	917.56	[825.84, 917.56]	872.23	Yes

*For the first four problems, $S = 25$ and $e^{-S} = 1.38879 \times 10^{-11}$. For the last three problems, $S = 30$ and $e^{-S} = 9.35762 \times 10^{-14}$. Again, $1-e^{-S}$ is practically one.

Again, in all cases the interval includes the optimum. Also, the intervals are fairly tight but not as tight as the Table 3 intervals. This is partly due to the fact that the sample size in Table 4 is smaller than it is in Table 3 (see Ref. [6] for details).

In addition to showing the analysis of a heuristic solution v to a particular combinatorial problem, Tables 3 and 4 provide a basis for comparing the accuracy of various heuristic procedures. For example, the heuristic solutions x_i to the 318-node problem were obtained by the Lin-Kernighan algorithm, which is a substantial improvement on the 2-opt heuristic. From Table 3, we can see that the performance measure $\tilde{b}/v$ (a measure of relative interval width) is generally smaller when the more powerful heuristic is used. The heuristic solutions x_i studied in Table 4 were obtained via the Clarke-Wright algorithm, which is not as powerful as the 2-opt procedure [7]. Again the performance measure $\tilde{b}/v$ reflects this fact. That is, in a relative sense the intervals are tighter when using the 2-opt procedure. Thus, the more powerful the heuristic, the tighter are the confidence intervals that can be expected.

CONCLUSIONS

In this paper, we have introduced a statistical approach for attacking intractable mathematical optimization problems. Straightforward and efficient procedures have been developed to perform interval estimation in the case where the globally optimal solution is unknown. The

interval-estimation procedure has been quite successful. The $100(1 - e^{-s})\%$ confidence intervals derived are of the simple form $[v - \tilde{b}, v]$.

Much work remains to be done. Interval-estimation construction on additional large traveling salesman test problems should be pursued, and, of course, we would like to see this technique applied to other combinatorial problems.

ACKNOWLEDGMENTS

We wish to thank Prof. H. Leon Harter for making his computer program available to us in this study. In addition, we thank Prof. Jim Yee for helpful comments. The research of the first author was supported in part by a University of Maryland General Research Board Faculty Research Award.

REFERENCES

[1] Beardwood, J., J. Halton, and J. Hammersley, "The Shortest Path Through Many Points," Proceedings of the Cambridge Philosophical Society *55*, 299-327 (1959).

[2] Clough, D., "An Asymptotic Extreme Value Sampling Theory for the Estimation of a Global Maximum," Canadian Operational Research Society Journal *7*, 102-115 (1969).

[3] Dannenbring, D., "Estimating Optimal Solutions for Large Combinatorial Problems," Management Science *23*, 1273-1283 (1977).

[4] Fisher, R., and L. Tippett, "Limiting Forms of the Frequency Distribution of the Largest or Smallest Member of a Sample," Proceedings of the Cambridge Philosophical Society *24*, 180-190 (1928).

[5] Garey, M., and D. Johnson, "Approximation Algorithms for Combinatorial Problems: An Annotated Bibliography," in *Algorithms and Complexity: New Directions and Recent Results*, J. Traub, ed. (Academic Press, New York, 1976), pp. 41-52.

[6] Golden, B., "A Statistical Approach to the TSP," Networks *7*, 209-225 (1977).

[7] Golden, B., "Point Estimation of a Global Optimum for Large Combinatorial Problems," Communications in Statistics *B7*, 361-367 (1978).

[8] Gumbel, E., *Statistics of Extremes* (Columbia University Press, New York, 1958).

[9] Harter, H., and A. Moore, "Maximum Likelihood Estimation of the Parameters of the Gamma and Weibull Populations from Complete and from Censored Samples," Technometrics *7*, 639-643 (1965).

[10] Held, M., and R. Karp, "The Traveling Salesman Problem and Minimum Spanning Trees," Operations Research *18*, 1138-1162 (1970).

[11] Held, M., and R. Karp, "The Traveling Salesman Problem and Minimum Spanning Trees: Part II," Mathematical Programming *1*, 6-25 (1971).

[12] Karp, R., "On the Computational Complexity of Combinatorial Problems," Networks *5*, 45-68 (1975).

[13] Klein, S., "Monte Carlo Estimation in Complex Optimization Problems," Doctoral Dissertation, The George Washington University, Washington, D.C. (1975).

[14] Krolak, P., W. Felts, and G. Marble, "A Man-Machine Approach Toward Solving the Traveling Salesman Problem," Communications of the Association for Computing Machinery (ACM) *14*, 327-334 (1971).

[15] Lin, S., "Computer Solutions of the TSP," Bell System Technical Journal *44*, 2245-2269 (1965).

[16] Lin, S., and B. Kernighan, "An Effective Heuristic Algorithm for the Traveling Salesman Problem," Operations Research *21*, 498-516 (1973).

[17] Mann, N., R. Schafer, and N. Singpurwalla, *Methods for Statistical Analysis of Reliability and Life Data* (John Wiley and Sons, New York, 1974).

[18] McRoberts, K., "A Search Model for Evaluating Combinatorially Explosive Problems," Operations Research *19*, 1331-1349 (1971).

[19] Mosteller, F. "Note on Application of Runs to Quality Control Charts," Annals of Mathematical Statistics *12*, 228-232 (1941).

[20] Nijenhuis, A., and H. Wilf, *Combinatorial Algorithms* (Academic Press, New York, 1975).

[21] Padberg, M., and S. Hong, "On the Symmetric Traveling Salesman Problem: A Computational Study," Working Paper No. 77-89, Graduate School of Business Administration, New York University, New York, N.Y. (1977).

[22] Robson, D., and J. Whitlock, "Estimation of a Truncation Point," Biometrika *51*, 33-39 (1964).

[23] Sahni, S., "General Techniques for Combinatorial Approximation," Operations Research *25*, 920-936 (1977).

[24] Stein, D., "Scheduling Dial-A-Ride Transportation Systems: An Asymptotic Approach," Doctoral Dissertation, Harvard University, Cambridge, Mass. (1977).

[25] Swed, F., and C. Eisenhart, "Tables for Testing Randomness of Grouping in a Sequence of Alternatives," Annals of Mathematical Statistics *14*, 66-87 (1943).

[26] Zanakis, S., "Computational Experience with Some Nonlinear Optimization Algorithms in Deriving Maximum Likelihood Estimates for the Three-Parameter Weibull Distribution," TIMS Studies in the Management Sciences *7*, 63-77 (1977).

LEAST-ABSOLUTE-VALUE ESTIMATORS FOR ONE-WAY AND TWO-WAY TABLES

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ABSTRACT

This paper concerns itself with the problem of estimating the parameters of one-way and two-way classification models by minimization of the sum of the absolute deviations of the regression function from the observed points. The one-way model reduces to obtaining a set of medians from which optimal parameters can be obtained by simple arithmetic manipulations. The two-way model is transformed into a specially structured linear programming problem, and two algorithms are presented to solve this problem. The occurrence of alternative optimal solutions in both models is discussed, and numerical examples are presented.

INTRODUCTION

An important problem in statistics is the study of the effect of one or two factors on a dependent variable. This problem can be formulated as a regression analysis using dummy (0,1) variables to represent the levels of the factors, and the resulting least squares analysis (LSQ) is well known [29]. Recently, the least-squares approach has come under considerable criticism, and several resistant estimation procedures have been proposed [1,19-22]. Along with the resistant estimation techniques has come an increased computational burden, and in some cases subjective decisions concerning outliers [14,27,31], weight functions [6], and score functions [21] must be made by the statistician. Minimizing the sum of the absolute values of the residuals is a robust procedure [19] which in some cases bypasses the latter difficulty. The computations involved in obtaining least-absolute-value (LAV) estimates have been a major deterrent to its use. This paper will demonstrate how LAV estimates can efficiently be obtained for one-way and two-way tables. A second difficulty in LAV estimation is the existence of alternate optimal solutions. For one-way and two-way tables alternative optimal solutions will *frequently* exist. However, a unique solution can be obtained by placing appropriate restrictions on the parameters. From a data-analytic point of view this may be regarded as an advantage since it requires some careful thought in the selection of additional contraints that will yield a "good" solution. In some situations this simple row-plus-column fit provides a first step in data analysis. The fitted values and residuals are used to identify possible outliers or to suggest how an improved fit may be obtained (see, e.g., [1], [6]). The LAV estimates will also provide a good starting point for resistant procedures that are iterative and require residuals from an initial fit to initiate the procedure.

Charnes, Cooper, and Ferguson [12] appear to be the first to present a practical approach to the solution of the general linear LAV problem. They demonstrated how the problem could be transformed into a linear programming problem and thus be solved by use of the well-developed theory of linear programming (LP). They also proved the statistical consistency of the estimates for LAV or any other norm-functional. Other papers primarily concerned with the use of LP to solve LAV problems are Refs. [2,4,5,28,30]. The main point to be gleaned from the more recent of these references is that a special-purpose primal simplex algorithm has proven to be the most efficient method for the solution of linear LAV problems. A reasonable alternative to the special-purpose primal algorithm is to take the dual of the original LP problem and solve it via an LP code with simple upper bounding. Section 3 outlines this transformation for a two-factor model while Wagner [32] gives a more detailed presentation for the general case. Computational results [3] indicate that the dual approach takes approximately four times as long as the special-purpose primal algorithm, but the algorithm for solving the dual has the definite advantage of being more widely available.

In Section 2, we demonstrate how LAV estimates can be obtained for a one-factor model without LP. In Section 3, two computer-oriented approaches for the analysis of a two-factor model using LP are presented. Both methods exploit the topological structure of LP problems to provide efficient solution techniques. Section 4 presents sufficient conditions for alternative optimal solutions to exist when additional criteria are not present. Examples illustrating LAV estimation for one-way and two-way tables are given in Section 5.

2. ONE-WAY CLASSIFICATION MODEL

Suppose it is hypothesized that observed values of a random variable are affected by t levels of a certain factor. A statistical model to study these effects may be stated as follows:

$$y_{ik} = \mu + \tau_i + e_{ik};\ i = 1, 2, \ldots, t;\ k = 1, 2, \ldots, n_i;$$

where y_{ik} is the k-th observation at the i-th level, μ is a typical value, τ_i is the effect associated with the i-th level, and e_{ik} is an unobservable random "error".

The LAV estimates for μ and τ_i, $i = 1, 2, \ldots, t$, by definition solve the following problem:

$$\text{(2.1)} \qquad \text{Minimize } z = \sum_{i=1}^{t} \sum_{k=1}^{n_i} |y_{ik} - (\mu + \tau_i)|.$$

An immediate difficulty arises because we have one degree of freedom in choosing values for the parameters; that is, μ or any one of the τ_i's may be assigned an arbitrary value without affecting the optimal value of z. The same difficulty arises in LSQ estimation, and it is averted by assuming that the total of the effects should be zero. Thus, the degree of freedom is absorbed by the constraint:

$$\text{(2.2)} \qquad \sum_{i=1}^{t} \tau_i = 0.$$

In a LAV analysis, this degree of freedom must also be removed by an additional constraint, but now the form of the constraint is not so obvious. To see this, consider the t disjoint problems:

$$\text{(2.3)} \qquad \text{Minimize } \sum_{k=1}^{n_i} |y_{ik} - \alpha_i|,\ i = 1, 2, \ldots, t,$$

where $\alpha_i = \mu + \tau_i$, $i = 1, \ldots, t$. An optimal value for α_i, say $\tilde{\alpha}_i$, is the median of the points y_{ik}, $k=1, 2, \ldots, n_i$. It then follows that, if we were using (2.2) as a constraint on the τ_i's, the optimal solution would be:

$$\tilde{\mu} = \left(\sum_{i=1}^{t} \tilde{\alpha}_i\right)/t \tag{2.4}$$

and

$$\tilde{\tau}_i = \tilde{\alpha}_i - \tilde{\mu}, \; i = 1, 2, \ldots t. \tag{2.5}$$

The $\tilde{\mu}$ given by (2.4) is the arithmetic mean of t medians. One reasonable alternative would be to choose $\tilde{\mu}$ to be the median of all observations, but, to parallel the LSQ analysis as closely as possible, we take a different approach. First note that (2.2) is equivalent to our taking up the degree of freedom by choosing μ and τ_i, $i=1, 2, \ldots, t$, so as to minimize

$$\sum_{i=1}^{t} \tau_i^2 = \sum_{i=1}^{t} (\alpha_i - \mu)^2$$

while still providing LSQ estimates. Correspondingly, to obtain parameters for the LAV estimate we minimize

$$\sum_{i=1}^{t} |\tau_i| = \sum_{i=1}^{t} |\alpha_i - \mu| \tag{2.6}$$

while maintaining the minimum value for z.

When the optimal value of α_i is unique (i.e., the median of the points y_{ik}, $k=1, 2, \ldots, n_i$, is unique) for all i, $\tilde{\mu}$ is the median of $\tilde{\alpha}_i$, $i=1, 2, \ldots, t$, and $\tilde{\tau}_i$ is obtained from (2.5). However, frequently the median of the y_{ik}'s is not unique but rather can lie anywhere within a continuous closed interval. When this is the case, $\tilde{\mu}$ can be obtained as follows:

STEP 1: Set U equal to the smallest lower bound of the intervals within which the optimal value of the α_i's must lie (unique values have the same upper and lower bound.)

STEP 2: Increase the value of U until any further increase would place more intervals completely below U than there are completely above U.

STEP 3: Place L equal to the current value of U.

STEP 4: Decrease the value of L until any further decrease would place more intervals completely above L than there are completely below L.

All the values in the closed interval $[L, U]$ are optimal for μ subject to the additional criterion (2.6). Let μ^* denote the median of all the y_{ik}'s and choose the point in $[L, U]$ that minimizes $|\mu - \mu^*|$. This criterion provides an estimate that is as close as possible to the estimate of μ that is obtained under the minimal one-parameter model (i.e., under the hypothesis that all the τ_i's are zero). A similar procedure will be used to obtain estimates of the parameters in the two-factor model (see Section 5). Once $\tilde{\mu}$ has been chosen from within this interval, $\tilde{\alpha}_i$ is chosen to be as close to $\tilde{\mu}$ as possible while remaining in the range of optimality for (2.3). The τ_i's are then determined from (2.5).

This LAV estimate is unique. Although the additional criteria that were added to force a unique solution are arbitrary, they are reasonable for this situation. Other approaches, similar

to the goal-programming (constrained regression) approach [8,9,11,24] proposed here, can be used to define a unique optimal solution, or we could let $\tilde{\mu} = (L + U)/2$. Unless these additional criteria are rather complex, LAV estimates are easily obtained for a one-way table. However, as we shall see in the next section, the extension of the LAV approach to two-way tables is far more complex than the corresponding LSQ extension.

3. TWO-WAY TABLE

3.1 Definition of Model

A two-factor model arises when a second factor is introduced as follows:

$$y_{ijk} = \mu + \tau_i + \beta_j + e_{ijk}, \quad i=1, \ldots, r; \quad j=1, \ldots, c; \quad k=1, \ldots, n_{ij}.$$

Thus, y_{ijk} is the k-th observation at the i-th level of the first factor and the j-th level of the second factor; τ_i represents the effect of the i-th level of the first factor (i.e., row effect), β_j represents the effect of the j-th level of the second factor (column effect), and μ is a typical value. LAV estimates of the parameters are obtained by solution of the following problem:

$$\text{(3.1)} \qquad \text{Minimize } z = \sum_{i=1}^{r} \sum_{j=1}^{c} \sum_{k=1}^{n_{ij}} |y_{ijk} - (\mu + \tau_i + \beta_j)|.$$

There are two degrees of freedom in the assignment of values to μ, τ_i, and β_j; thus, restrictions should be added to the problem. In the corresponding LSQ analysis, (2.2) and

$$\text{(3.2)} \qquad \sum_{j=1}^{c} \beta_j = 0$$

are appended. This is equivalent to the provision of LSQ estimates which minimizes

$$\sum_{i=1}^{r} \tau_i^2 + \sum_{j=1}^{c} \beta_j^2.$$

Analogously, in the LAV analysis we provide estimates that minimize

$$\text{(3.3)} \qquad \sum_{i=1}^{r} |\tau_i| + \sum_{j=1}^{c} |\beta_j|$$

subject to the optimal value for z in (3.1) being maintained. As in the one-way analysis, this additional criterion does not necessarily provide a unique solution (see Section 5 for an alternative). Further restrictions, or a completely different set of criteria, may determine a unique solution. Our purpose is to present what we feel are reasonable conditions for LAV estimates to satisfy.

3.2 Computational Approaches

Before (3.3) is considered, two computer-oriented approaches used to obtain an optimal solution to (3.1) will be discussed. We again make the transformation $\alpha_i = \mu + \tau_i$, and restate (3.1) as:

$$\text{(3.4)} \qquad \text{Minimize } z = \sum_{i} \sum_{j} \sum_{k} |y_{ijk} - (\alpha_i + \beta_j)|.$$

Problem (3.4) can be written as a linear programming problem:

(3.5) $$\text{Minimize} \sum_i \sum_j \sum_k (d_{ijk}^+ + d_{ijk}^-)$$

subject to

$$\alpha_i + \beta_j - y_{ijk} + d_{ijk}^+ - d_{ijk}^- = 0,$$

$$d_{ijk}^+ \geqslant 0,\ d_{ijk}^- \geqslant 0,$$

$$i=1, \ldots, r;\ j=1, \ldots, c;\ k=1, \ldots, n_{ij},$$

where d_{ijk}^+ and d_{ijk}^- are the positive and negative deviations of the regression equation from the observation y_{ijk}, respectively. Problem (3.5) is not tractable in its present form for a direct application of the simplex algorithm. The main reason for this is that the number of constraints is equal to the number of observations, which may give rise to an excessively large basis matrix. This difficulty can be overcome by taking the dual of (3.5), which is given by:

(3.6) $$\text{Maximize} \sum_i \sum_j \sum_k y_{ijk}\, \pi_{ijk}$$

subject to

$$\sum_j \sum_k \pi_{ijk} = 0,\ i=1, \ldots, r;$$

$$\sum_i \sum_k \pi_{ijk} = 0,\ j=1, \ldots, c;$$

$$-1 \leqslant \pi_{ijk} \leqslant 1,\ i=1, \ldots, r;\ j=1, \ldots, c;\ k=1, \ldots, n_{ij}.$$

By making the transformation $\pi'_{ijk} = \pi_{ijk} + 1$, we can write (3.6) in a more-standard linear programming format:

(3.7) $$\text{Maximize} \sum_i \sum_j \sum_k (y_{ijk}\, \pi'_{ijk} - y_{ijk})$$

subject to

$$\sum_j \sum_k \pi'_{ijk} - \sum_j n_{ij},\ i=1, \ldots, r,$$

$$\sum_i \sum_k \pi'_{ijk} - \sum_i n_{ij},\ j=1, \ldots, c,$$

and

$$0 \leqslant \pi'_{ijk} \leqslant 2,\ i=1, \ldots, r;\ j=1, \ldots, c;\ k=1, \ldots, n_{ij}.$$

It can now be recognized that (3.7) is a capacitated transportation problem [10] with r origins and c destinations except that, because of multiple observations within cells, there is more than one path from origin i to destination j. We can incorporate this extension into the standard *LP* algorithm by considering π'_{ijk}'s for entry into the basis only when all other *LP* variables corresponding to observations in cell (i, j) with a value larger than y_{ijk} are at their upper bound. Computational results [15,16] indicate that transportation problems can be solved approximately 150 times faster by using a special-purpose primal simplex code as opposed to a general-purpose state-of-the-art LP code. Thus, considerable savings can be derived if we recognize the special structure of (3.6).

Once (3.6) has been solved, optimal values for the α_i's and β_j's in (3.4) are given by the dual variables or simplex multipliers for the first $r + c$ constraints. There is, however, one degree of freedom in the choice of the α_i's and β_j's, and a second degree of freedom in the assignment of values to the τ_i's and μ. These degrees of freedom can be taken up if we satisfy criterion (3.3). We delay the discussion of how to accomodate (3.3) until after the primal approach to (3.5) has been presented.

For the general problem of obtaining parameters for absolute-deviations estimates, it has been shown [3,4,30] that solving the general-case equivalent of (3.5) directly with a special-purpose primal algorithm is computationally the most efficient approach available. There is no reason to believe that this would not also be true here as the structure of the problem can still be utilized to perform the operations of the algorithm without a matrix ever being inverted explicitly. This algorithm, modified to take advantage of (3.5)'s structure, will not be developed here, but a brief overview is given to indicate the use of techniques found in solving transportation problems and to state a formula required in the next section.

We begin by restating the constraints of (3.5) in matrix notation as follows:

$$A\gamma - Y + D^+ - D^- = 0, \tag{3.8}$$

$$D^+ \geqslant 0,\ D^- \geqslant 0, \tag{3.9}$$

where $\gamma' = (\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_c)$, and A is a $\left(\sum_i \sum_j n_{ij}\right)$ by $(r + c)$ matrix of 0's and 1's with a single dependent column. It is clear from the objective function of (3.5) that (3.8) can also be written as:

$$Y - D^+ \leqslant A\gamma \leqslant Y + D^-. \tag{3.10}$$

The algorithm at any stage works with a basis consisting of $(r + c - 1)$ rows of A. To distinguish between the basic and nonbasic rows of A we partition A, Y, D^+, and D^- and rewrite (3.10) as

$$\begin{bmatrix} Y_b \\ Y_n \end{bmatrix} - \begin{bmatrix} D_b^+ \\ D_n^+ \end{bmatrix} \leqslant \begin{bmatrix} B \\ N \end{bmatrix} \gamma \leqslant \begin{bmatrix} Y_b \\ Y_n \end{bmatrix} + \begin{bmatrix} D_b^- \\ D_n^- \end{bmatrix},$$

where B denotes the basis and N the remaining rows of A. If we let $\lambda = B\gamma$, then the constraints (3.8) become

$$Y_b - D_b^+ \leqslant \lambda \leqslant Y_b + D_b^-$$

$$Y_n - D_n^+ \leqslant NB^{\#}\lambda \leqslant Y_n + D_n^-,$$

where $B^{\#}$ is a generalized inverse of B. The current solution is $\lambda^* = Y_b$, $D_b^+ = D_b^- = 0$, $\gamma^* = B^{\#}\lambda^*$, and the deviations in the nonbasic rows are as small as possible based on λ^* and the constraints. The structure of B allows it to be stored as a spanning tree [15] similar to that of the basis of a transportation problem. This allows us to use the triangularity of B (after a dependent column is dropped) to solve $B\gamma = \lambda$ without our explicitly obtaining a $B^{\#}$. Thus, $B^{\#}$ is not required to obtain a basic solution; and, in fact, it is never needed.

The next step in the algorithm is to determine whether on not an increase or decrease in any λ_i away from its current value λ_i^* will decrease the objective value. The objective function rate of change is $1 + \theta_i$ or $1 - \theta_i$ when λ_i is increased or decreased, respectively, where $\theta = (\theta_1, \theta_2, \ldots, \theta_{r+c-1})$ is given by

$$\theta = \bar{\sum_j} N_j B^{\#} - \stackrel{+}{\sum_j} N_j B^{\#} = \left(\bar{\sum_j} N_j - \stackrel{+}{\sum_j} N_j\right) B^{\#} \tag{3.11}$$

or

$$\theta B = \sum_{j}^{-} N_j - \sum_{j}^{+} N_j.$$

In (3.11) the $^+$ and $^-$ superscripts indicate summation over rows of N with positive and negative deviations, respectively. When a nonbasic row has a zero residual it is classified by the algorithm as a positive or negative zero; thus, every nonbasic row will be included exactly once in the above summation. Again, the triangularity of B allows us to obtain θ without $B^{\#}$ (just as in the transportation algorithm a primal solution is obtained). Equality (3.11) will be important in the next section, where conditions for alternative solutions to the LP problem (3.5) are discussed.

Since we are minimizing the sum of the absolute deviations, a basis change would be made if $|\theta_i| > 1$. Hence, the current solution is recognized as optimal when $|\theta_i| \leqslant 1$, $i = 1, 2, \ldots, r + c - 1$.

The pivot rule of Barrodale and Roberts [4], which may combine several standard simplex pivots into one, is used to determine the row of N to enter the basis. The details of implementing this rule while effectively utilizing the structure of B and N will not be given here, but the major computational step is similar to calculating the ratio in a dual simplex transportation algorithm [17].

3.3 A Secondary Criterion for Choosing the Parameter Estimates

The remainder of this section will be devoted to describing how the secondary criterion (3.3) can be considered within the framework of an LP algorithm. The procedure can be utilized on a revised version of (3.5) or (3.7), and it is similar to the perturbation method of Charnes [7]. Like Charnes' method an arbitrarily small positive number ϵ will be used in the description, but the most efficient implementation would never assign a value to ϵ and all calculations involving ϵ are performed implicitly. However, if we place ϵ equal to a specific value additional computer coding is avoided. We begin by noting that (3.3) can be expressed in LP form as:

$$\text{(3.12)} \qquad \text{Minimize} \sum_{j=1}^{r+c} (\delta_j^+ + \delta_j^-),$$

subject to

$$\tau_i + \delta_i^+ - \delta_i^- = 0, \quad i = 1, \ldots, r,$$

$$\text{(3.13)} \qquad \beta_j + \delta_{r+j}^+ - \delta_{r+j}^- = 0, \quad j = 1, \ldots, c,$$

$$\delta_j^+ \geqslant 0 \text{ and } \delta_j^- \geqslant 0, \quad j = 1, \ldots, r + c,$$

where δ_j^+ and δ_j^- are the positive and negative deviations of the effects from zero.

Problem (3.12) is of only secondary concern, since we wish always to obtain an optimal solution to (3.5). The desired optimal solution is given in a limiting sense ($\epsilon \to 0$) by solving

$$\text{(3.14)} \qquad \text{Minimize} \sum_{i} \sum_{j} \sum_{k} (d_{ijk}^+ + d_{ijk}^-) + \sum_{j=1}^{r+c} (\epsilon\delta_j^+ + \epsilon\delta_j^-)$$

subject to the constraints (3.8), (3.9), and (3.13).

Problem (3.14) is not in the form in which the columns have the exact structure that the rows of a transportation problem possess. To obtain the desired format let $\beta_{c+1} = -\mu$ and create a "dummy parameter" β_{c+2} (this is a variable in the LP problem). The problem then becomes:

$$\text{(3.15)} \qquad \text{Minimize} \quad \sum_i \sum_j \sum_k (d_{ijk}^+ + d_{ijk}^-) + \sum_{j=1}^{r+c} (\epsilon\delta_j^+ + \epsilon\delta_j^-),$$

subject to

$$\alpha_i + \beta_j - y_{ijk} + d_{ijk}^+ - d_{ijk}^- = 0, \quad i = 1, \ldots, r, \quad j = 1, \ldots, c, \quad k = 1, \ldots, n_{ij},$$

$$\alpha_i + \beta_{c+1} + \delta_i^+ - \delta_i^- = 0, \quad i = 1, \ldots, r,$$

$$\beta_{c+2} + \beta_j + \delta_{r+j}^+ - \delta_{r+j}^- = 0, \quad j = 1, \ldots, c,$$

$$\delta_j^+ \geqslant 0, \ \delta_j^- \geqslant 0, \ d_{ijk}^+ \geqslant 0, \ d_{ijk} \geqslant 0$$

where the degree of freedom in assigning values to the parameters is absorbed by always placing $\beta_{c+2} = 0$.

The algorithm described to solve (3.5) directly can be used to solve (3.15) with a slight modification to account for a weight of ϵ, rather than one, on the deviations of τ_i and β_j away from zero. Also, if we take the dual of (3.15) and make the lower bound on the variables in this dual problem zero, a capacitated transportation problem is again created. This can, of course, be solved with a standard code; however, we must take care to ensure that $\beta_{c+2} = 0$ when working back to the optimal solution to (3.15).

The formulation just described takes care of the two degrees of freedom at the expense of creating an additional "source" and an additional "destination" in the transportation problem, and the possibility of alternative optimal solutions has been reduced considerably. The problem of alternative optimal solutions to (3.3) is discussed in the next section along with statements of sufficient conditions for alternative optimal solutions to exist.

4. ALTERNATIVE OPTIMAL SOLUTIONS

A disturbing aspect of LAV estimation for two-way tables is that alternative optimal solutions frequently occur, and decidedly different estimates are obtainable. This difficulty may be averted if we specify additional criteria for the estimates to satisfy. It is the purpose of this section to indicate that alternative optimal "fits" are to be expected in the analysis of two-way tables via LAV procedures if (3.1) is the sole criterion. This serves to emphasize the importance of "good" additional criteria.

It is well known that the median of an even number of observations is unique only when the two middle observations have the same value. We obtain the parameters for the one-way model (2.3) by taking the median of t sets of observations, and the values will be unique only when all t medians are unique. However, we can always obtain a unique solution by adding the additional restrictions described in Section 2.

With respect to the two-way model, it can be shown that an LP solution to (3.5) is optimal if the θ we obtain by solving (3.11) satisfies

$$-1 \leqslant \theta_i \leqslant 1, \quad i = 1, 2, \ldots, r + c - 1.$$

Furthermore, the basis B is a *unique* optimal basis only when

$$-1 < \theta_i < 1,\ i = 1, 2, \ldots, r + c - 1;$$

in other words, an alternative optimal basis exists if at the completion of the algorithm θ_i equals -1 or $+1$ for any i. But because N_j is a vector of 0's and 1's, and because B is a unimodular matrix [18], θ will always have integer components. Hence, at optimality θ_i will equal either -1, $+1$ or 0. This means that the current optimal basis is unique if and only if all the components of θ are zero, and this will only be true when

$$\sum_{j}^{-} N_j - \sum_{j}^{+} N_j = 0. \tag{4.1}$$

Condition (4.1) forms the foundation for proving the theorem of this section. It might be well to point out at this time that all our results relate only to alternative optimal *basic matrices*, not to alternative optimal fits. However, an alternative optimal basis is equivalent to an alternative optimal fit whenever an optimal fit interpolates exactly $r + c - 1$ points. Theoretically, for fixed sample size this will occur with probability 1 whenever the observations are taken from a continuous population.

The following theorem is concerned with the special case of a two-way classification model where $n_{ij} = 1$ for all i and j.

THEOREM 4.1: The LP problem (3.3), which is equivalent to the problem of finding LAV estimates for a two-way classification model with exactly one observation per cell, will have alternative optimal basic matrices whenever the minimum of r and c is even.

PROOF: Suppose that the LP problem has been solved and an optimal basic matrix has been obtained. For this matrix to be a unique optimal basic matrix, condition (4.1) must be satisfied. This occurs only if for each nonzero component from an N_j associated with a positive deviation there corresponds a nonzero component from an N_j associated with a negative deviation. In other words, the nonbasic rows of A must contain an even number of nonzero coefficients in each column, because the sum of an odd number of plus or minus ones will never be zero. The proof of the theorem will consist of showing that, whenever the minimum of c or r is even, there is a least one column with an odd number of nonzero components (+1's) in the nonbasic rows.

For explanatory purposes, we assume $r \geqslant c$, but the proof follows in an analogous manner if the reverse is true. There are $r + c - 1$ rows of A in the basis B and, because B forms a basis, every column of B must have at least one nonzero entry. We note that B is a submatrix of A and that each row has one and only one nonzero entry in the first r components and one and only one nonzero entry in the last c components. Also, each of the last c columns and A has exactly r 1's with the remaining coefficients being 0's. To satisfy (4.1) there must be an even number of 1's in each of the last c columns of N. Thus, since r is even, there must be an even number of 1's in these c columns of B. But each column of B must have at least one nonzero entry, and then there must be at least two 1's present. This would require B to have at least $2c > c + r - 1$ rows. Therefore, at least one of the last c columns of B has a single nonzero entry. The proof of the theorem now follows from the inability to satisfy (4.1).

It is not difficult to derive examples of two-way tables with a unique optimal basis for the LP equivalent. Consider the two-way table of Exhibit 1. This example has a unique optimal basis matrix with the optimal fit interpolating observations 3, 4, 5, 6, and 7. However,

EXHIBIT 1. *Two-way table with a unique optimal LP basis matrix.*

5	6	7
4	8	1
3	2	9

alternative optimal basic matrices exist if the 6 and 8 interchange position. Certainly our computational experience would indicate that, even if the conditions of Theorem 4.1 are not satisfied, alternative optimal fits are more likely to appear than not.

Furthermore, a unique optimal basis matrix does not clearly define the estimates for the parameters. There are two degrees of freedom that provide us with the ability to arbitrarily choose values for two parameters and remain optimal. In the previous sections we have proposed additional criteria that deal with this problem, and we will discuss this matter further in the next section via numerical examples.

5. NUMERICAL EXAMPLES

5.1 LAV Analysis of One-Way Table

To illustrate the application of the algorithm described in Section 2, we will use the Nebraska voting data shown in Exhibit 2 [31, Chap. 19]. In this section two separate one-way analyses will be carried out. These results are used in Section 5.3, where the same data is used to illustrate the LAV analysis of a two-way table.

EXHIBIT 2. *Nebraska Voting—Raw % Democratic for 11 Counties in 12 Presidential Elections (unit = 0.1%)*

County	Year											
	'20	'24	'28	'32	'36	'40	'44	'48	'52	'56	'60	'64
D0	353	358	589	757	544	365	345	337	189	167	236	396
D1	323	252	236	669	396	267	238	257	149	148	138	290
B1	288	302	305	619	510	397	372	411	234	268	279	389
D2	379	372	270	606	497	363	388	433	196	223	251	399
D4	342	226	264	626	510	407	404	496	230	264	222	374
B4	270	291	247	569	450	354	325	374	218	259	229	410
D5	228	177	150	553	426	349	272	472	177	240	225	336
B5	270	237	227	561	425	352	340	360	179	232	189	310
D6	265	196	165	547	472	336	313	436	195	219	226	388
B7	322	257	454	661	513	384	379	454	253	307	370	462
D7	270	191	352	776	526	463	442	553	337	358	360	439

Source: Tukey, J.W., *Exploratory Data Analysis, II* (Addison-Wesley, Reading, Massachusetts, 1971).

First, we consider the rows (i.e., counties). The median intervals are shown in Exhibit 3, and, using the algorithm in Section 2, we obtain $(L, U) = (325, 342)$. Since the median $\mu^* = 338$, we set $\mu = 338$ and obtain the fitted values and effects shown in the last two columns of Exhibit 3. The same procedure is then applied to the columns (i.e., years) and the fitted values and effects are shown in Exhibit 4.

EXHIBIT 3. *LAV One-Way Analysis of Counties (i.e., Rows of Exhibit 2) for the Nebraska Voting Data*

County	Median Interval	Fit	Effect
D5	(240,272)	272	−66
D1	(252,257)	257	−81
D6	(265,313)	313	−31
B5	(270,310)	310	−28
B4	(291,325)	325	−13
B1	(305,372)	338	0
D4	(342,374)	342	4
D0	(353,358)	353	15
D7	(360,439)	360	22
D2	(372,379)	372	24
B7	(379,384)	379	41

Note: The order has been changed to illustrate the procedure for obtaining the interval $(L,U) = (325,342)$.

EXHIBIT 4. *LAV One-Way Analysis of Years (i.e., Columns) for the Nebraska Voting Data*

Year	'20	'24	'28	'32	'36	'40	'44	'48	'52	'56	'60	'64
Effect	−50	−86	−74	281	159	25	7	95	−142	−98	−109	51
Fit	288	252	264	619	497	363	345	433	196	240	229	389

In the LSQ analysis the best estimates of the row and column effects (along with the overall mean) provided the solution to the two-way analysis. For the LAV analysis this is not true, but in Section 5.3 we propose obtaining the LAV estimates for the two-way table that are as close as possible to these restricted fits. It will then be possible to assess the relative importance of the row effects after column effects have already been included in the model.

5.2 Some Small Examples for Two-Way Tables

To illustrate the difficulties that arise in choosing values for the parameters in the two-way model, we consider the table given by Exhibit 5. We begin the analysis by obtaining LAV estimates with additional restrictions $\tau_1 + \tau_2 = 0$ and $\beta_1 + \beta_2 = 0$. We can obtain optimal parameter values from the LP by considering any extreme point that defines a hyperplane passing through three of the four observations. Thus, four optimal extreme-point solutions are possible, and they are given by Exhibit 6. All optimal solutions are given by convex combinations of these four points. Clearly, a great deal of discrepancy is possible among optimal solutions. If we consider only extreme-point solutions, the observation which the hyperplane does not interpolate will have a residual with absolute value 998 and the other points will, of course, have a zero residual. Thus, the LP solution could indicate that any one of the four observations has an unduly large residual and could make it a candidate for consideration as an outlier.

If we perform the LAV analysis on the same table with additional criterion (3.3) rather than (2.2) and (3.2), a unique solution ($\mu = 1$, $\tau_1 = \tau_2 = \beta_1 = \beta_2 = 0$) is obtained. An inspection of the residuals now indicates that the observation in cell (2, 2) might be considered an outlier.

EXHIBIT 5. *Sample Two-Way Table with a Single Outlier*

1	1
1	999

EXHIBIT 6. *Optimal Extreme-Point Solutions to the Two-Way Table of Exhibit 5 with the Constraints* $\tau_1 + \tau_2 = 0$ *and* $\beta_1 + \beta_2 = 0$ *Added*

Extreme-Point Solution	μ	τ_1	τ_2	β_1	β_2
1	500	−499	499	0	0
2	1	−499	499	−499	499
3	500	0	0	−499	499
4	1	0	0	0	0

The two-by-two table of Exhibit 5 is an extreme case, and it was presented to indicate what could occur in the LAV analysis of two-way tables if caution is not exercised. Generally, such widely divergent solutions will not be encountered, regardless of the additional criteria employed to absorb the degrees of freedom.

The next example (Exhibit 7) is a four-by-four table from Tukey [31, Chap. 22]. With the additional criterion (3.3) appended to the problem, eight optimal extreme point solutions were found. These are given in Exhibit 8. The complete set of optimal extreme point solutions can be generated [18, p. 166]; however, the amount of work required to do so is generally prohibitive. No attempt was made to generate all optimal extreme solutions to the two-way table in Exhibit 7.

EXHIBIT 7. *Sample Two-Way Table from Tukey [31, Chap. 22]*

718	732	734	793
725	781	725	716
704	1035	763	758
726	765	738	761

EXHIBIT 8. *Table of Eight Optimal Extreme-Point Solutions for the LAV Estimation Problem Obtained from Exhibit 7 with Criterion (3.3) Added*

Solution	μ	τ_1	τ_2	τ_3	τ_4	β_1	β_2	β_3	β_4
1	738	−4	−1	0	0	−12	27	0	20
2	742	−8	−1	0	0	−16	23	0	16
3	758	−4	−3	0	0	−30	7	−20	0
4	758	−8	−1	0	0	−32	7	−16	0
5	758	−4	0	0	0	−33	7	−20	0
6	738	−4	0	0	0	−13	27	0	20
7	741	−7	0	0	1	−16	23	0	17
8	758	−7	0	0	1	−33	6	−17	0

We note that all the solutions of Exhibit 8 indicate that the row effects are small relative to the column effects. Also, the residual for the outlier of cell (3,2) is 270 for all but the last solution, where it is 271.

5.3 LAV Analysis of a Two-Way Table

In Section 5.1 we considered the one-way analysis of the Nebraska voting data. The two-way analysis of this data by LSQ is shown in Exhibit 9. The LAV estimates are obtained if we solve (3.1) with the additional criterion that we minimize

$$(5.1) \qquad |\mu - \mu^*| + \sum_{i=1}^{r} |\tau_i - \tau_i^*| + \sum_{j=1}^{c} |\beta_j - \beta_j^*|$$

Subject to the optimal value for z in (3.1) being maintained. In (5.1), the superscript * denotes the LAV estimates that are obtained for the one-way fits (see Section 5.1). The robust elementary analysis we obtained using LAV is shown in Exhibit 10. The stem- and-leaf plots, hinges, midspreads, and side values for the residuals obtained from the LSQ and LAV fits are shown in Exhibit 11, and the large (outside, i.e., past the side values) residuals are identified in Exhibit 12.

EXHIBIT 9. *Elementary Analysis by Means (i.e., LSQ) of the Nebraska Voting Data (Unit = 0.1%)*

County	Year												Eff	Fit
	'20	'24	'28	'32	'36	'40	'44	'48	52	'56	'60	'64		
D0	15	61	225	88	27	−40	−40	−117	−63	−115	−49	−23	38	387
D1	91	61	8	106	−14	−31	−41	−91	3	−28	−41	−23	−69	280
B1	−29	26	−7	−28	15	14	9	−21	4	8	16	−8	16	365
D2	62	96	−42	−41	2	−20	25	0	−34	−37	−13	2	16	365
D4	26	−49	−47	−20	16	25	42	64	0	5	−41	−22	15	364
B4	−15	47	−33	−46	−13	3	−6	−27	20	31	−3	45	−15	334
D5	−25	−35	−98	−30	−5	30	−27	104	11	44	26	3	−48	301
B5	11	19	−27	−28	−12	27	35	−15	7	30	−17	−29	−42	307
D6	−0	−28	−96	−49	29	5	2	55	16	11	14	42	−36	313
B7	−31	−55	105	−23	−19	−36	−21	−15	−14	10	70	28	53	402
D7	−104	−142	−18	71	−26	23	21	63	49	40	39	-16	73	422
Eff	−48	−89	−53	283	130	18	−2	68	−135	−105	−101	32	349	0
Fit	301	260	296	632	479	367	347	417	214	244	248	381	0	−349

EXHIBIT 10. *Nebraska Voting Data from Exhibit 2 ROBUST Elementary Analysis by LAV (Unit = 0.1%)*

County	Year												Eff	Fit
	'20	'24	'28	'32	'36	'40	'44	'48	52	'56	'60	'64		
D0	56	81	322	144	50	−3	0	−83	−30	−85	−13	0	7	345
D1	124	73	67	154	0	−3	−9	−65	28	−6	−13	−8	−91	247
B1	−25	9	22	−10	0	13	11	−25	−1	0	14	−23	23	361
D2	79	92	0	−10	0	−8	40	10	−26	−32	−1	0	10	348
D4	33	−63	−15	1	4	27	47	64	−1	0	−39	−34	19	357
B4	−7	34	0	−24	−24	6	0	−26	19	27	0	34	−13	325
D5	−9	−40	−57	0	−8	41	−13	112	18	48	36	0	−53	285
B5	13	0	0	−12	−29	24	35	−20	0	20	−20	−46	−33	305
D6	0	−49	−70	−34	10	0	0	48	8	−1	9	24	−25	313
B7	−9	−54	153	14	−15	−18	0	0	0	21	87	32	41	379
D7	−122	−181	−10	68	−63	0	2	38	23	11	16	−52	102	440
Eff	−48	−68	−78	268	149	23	0	75	−126	−93	−96	51	338	0
Fit	290	270	260	606	487	361	338	413	212	245	242	389	0	−338

EXHIBIT 11. *Analysis of the 132 Residuals Obtained from the LSQ and LAV Two-Way Analyses.*

LSQ			LAV		
4	L	LLLL	2	L	LL
7	−9	751		−9	
	−8		4	−8	35
	−7		5	−7	0
8	−6	3	8	−6	335
9	−5	4	11	−5	472
19	−4	9771580911	14	−4	096
28	−3	043914957	19	−3	40294
47	−2	9487809815062772229	29	−2	5449560603
60	−1	3932169543375	39	−1	0002333558
(7)	−0	7360038	64	−0	79900008338900110601180000
65	0	0335923368524	(20)	0	09001004600002080009
52	1	51516749161254	48	1	34039098946
33	2	669573526	37	2	012344778
29	3	0151090	28	3	3458642
22	4	7255153	21	4	10788
15	5	5	16	5	60
14	6	21143	14	6	784
9	7	21	11	7	93
7	8	9	9	8	17
6	9	16	7	9	2
4	H	HHHH	6	H	HHHHHH

High	Low
256	−104
107	−142
107	−117
104	−114

High	Low
322	−181
154	−122
153	
144	
124	
112	

−27h	and	+26h	Hinges	−13	and	+23h
53h			Midspread	36h		
−81	and	+79	Side Values	−49h	and	60
7	and	7	Number Outside	11	and	14

EXHIBIT 13. *Elementary Analysis Using Pomedian Procedure on the Nebraska Voting Data (see Tukey [3, Exhibit 10, Chap. 19]).*

County	Year												Eff	Fit
	'20	'24	'28	'32	'36	'40	'44	'48	'52	'56	'60	'64		
D0	45	91	314	147	59	−14	−9	−73	−36	−95	−19	8	7	356
D1	105	75	53	149	1	−22	−26	−63	14	−2	−27	−8	−83	266
B1	−29	26	21	0	16	9	9	−8	0	−3	14	−8	16	365
D2	62	96	−19	−13	3	−25	−25	14	−38	−48	−13	2	16	365
D4	26	−49	−19	8	17	20	42	78	−3	−6	−41	−22	15	364
B4	−9	51	−1	−14	−8	2	−2	−9	20	24	1	49	−20	329
D5	−24	−34	−69	−1	−3	26	−26	118	8	34	26	4	−49	300
B5	9	13	−1	−2	−13	20	33	−3	1	17	−19	−31	−40	309
D6	0	−28	−67	−20	30	0	2	41	13	0	14	43	−36	313
B7	−30	−54	135	7	−16	−39	−19	0	−16	1	71	30	53	402
D7	−124	−162	−11	24	−45	−2	2	55	22	11	19	−35	93	442
Eff	−48	−89	−81	254	129	23	−2	54	−133	−94	−101	32	349	0
Fit	301	260	268	603	478	372	247	403	216	255	248	381	0	−349

analysis is essentially exploratory in nature. We are, on the one hand, prepared to obtain evidence that a simpler model may adequately fit the data. At the same time, we are ready to look at the residuals from a robust row-plus-column fit using diagnostic plots or other techniques that could suggest that outliers are present, or that an improved fit may be obtained if we add terms to the model or reexpress the data through a transformation.

6. DISCUSSION

It is generally recognized that after we obtain a simple row-plus-column fit for a two-way table the residuals should be carefully analyzed. Gentleman and Wilk [13] have considered the effect of one or two outliers superimposed on a basic additive model with independent normal fluctuations, with mean zero, and with constant variance. Their results indicate that when one outlier is present the judicious use of half-normal plotting provides a complete basis for data-analytic judgements. They further find that direct analysis of residuals (from a LSQ fit) is not reliably indicative of the existence of peculiarities when two outliers are present. Gentleman and Wilk [14] have also considered the problem of multiple outliers, and they have proposed methods for identification of the "K most likely outlier subset" (where the maximum possible value of K must be known). Their approach considers to what extent a p-parameter model analysis can be statistically improved by selective reduction in the size (n) of the data. Their method results in a LSQ analysis of the "good data."

In many situations the form of the model is only tentative, and a diagnostic plot of residuals is required [31]. The diagnostic plot may suggest that an improved fit can be obtained either by adding to the model or by reexpressing the y's. When there are several possible departures from the ideal additive model for a two-way table, the importance of obtaining a robust fit is increased if attempts to improve upon the conventional LSQ analysis are to be successful. Thus, as McNeil and Tukey [26] have shown, it is possible to begin with a simple row-plus-column fit of a two-way table using both LSQ and LAV. If the unknown e_{ij}'s follow a Gaussian distribution, then we expect that the residuals from both fits should appear to be near Gaussian, with somewhat less-stretched tails for the least-square residuals. If the e_{ij}'s are from a tail-stretched distribution, the residuals should be tail-stretched — the LSQ residuals much less than the e_{ij}'s and the LAV residuals slightly more. Tail-stretched residual distributions may also be the result of an inadequate model. Consequently, if the LSQ and LAV analyses are clearly different then further careful analysis is required. It is important to note, as Mallows

EXHIBIT 12. *Outside Residuals from Row-Plus-Column Fits Using LSQ (see Exhibits 9 and 11) and LAD (see Exhibits 10 and 11)*

County	Year											
	'20	'24	'28	'32	'36	'40	'44	'48	'52	'56	'60	'64
LSQ												
D0	.	.	255	88	.	.	.	−117	.	−115	.	.
D1	91	.	.	106	.	.	.	−91	.	.	.	.
B1	.	.	.	.	.	.	.	.	.	.	.	.
D2	.	96	.	.	.	.	.	.	.	.	.	.
D4	.	.	.	.	.	.	.	.	.	.	.	.
B4	.	.	.	.	.	.	.	.	.	.	.	.
D5	.	.	−98	.	.	.	.	104	.	.	.	.
B5	.	.	.	.	.	.	.	.	.	.	.	.
D6	.	.	−96	.	.	.	.	.	.	.	.	.
B7	.	.	105	.	.	.	.	.	.	.	.	.
D7	−104	142	.	.	.	.	.	.	.	.	.	.
LAV												
D0	.	81	322	144	.	.	.	−83	.	−85	.	.
D1	124	73	67	154	.	.	.	−65	.	.	.	.
B1	.	.	.	.	.	.	.	.	.	.	.	.
D2	79	92	.	.	.	.	.	.	.	.	.	.
D4	.	−63	.	.	.	.	.	64	.	.	.	.
B4	.	.	.	.	.	.	.	.	.	.	.	.
D5	.	.	−57	.	.	.	.	112	.	.	.	.
B5	.	.	.	.	.	.	.	.	.	.	.	.
D6	.	.	−70	.	.	.	.	.	.	.	.	.
B7	.	−54	153	.	.	.	.	.	.	.	87	.
D7	−122	−181	.	68	−63	.	.	.	.	.	.	.

Tukey [31, Chap. 19] obtained a resistant elementary analysis of this data using pomedian polishing on the LSQ analysis (see Exhibit 13). The pomedian procedure leads to residuals that are "nearly balanced" in sign in *each* row and column. Note that the median of each row and column of the LAV residuals in Exhibit 10 is zero.

5.4 Quality of Fit for Two-Way Table

In a LSQ analysis of a two-way table the importance of the row and column effects is measured in terms of the decrease in the sum of squares that occurs when the row (or column) effects are included in the model. For the LAV analysis it is also possible to obtain an indication of the importance of the row and column effects. First, we obtain $z(\mu^*) = \sum_i \sum_j |y_{ij} - \mu^*| = 13661$. Next, we calculate $z(\mu^*, \beta^*) = 6282$, $z(\mu^*, \tau^*) = 12531$, and $z(\tilde{\mu}, \tau, \beta) = 4240$. Then, using an approach suggested by McNeil and Tukey [26], we determine that the column fit accounts for $100[1 - (6282/13661)^2] = 78.9\%$ of the total variation, measured on a size-squared scale in terms of the sum of the absolute deviations. Similarly, 15.9% of the total variation is explained by the row fit, and the row-plus-column fit accounts for 90.4% of the total variation. Thus, we are able to conclude that the size of the residuals is considerably reduced if both row and column effects are included in the model.

As is the case in an unbalanced LSQ analysis, the reduction in the objective function that occurs when additional parameters are added to the model is order dependent. Consequently, this heuristic approach to evaluation of the relative importance of a given subset of parameters is similar to the use of r^2 values in the LSQ analysis. This approach is suggested when the

[25] has pointed out, that our understanding of robust techniques and the behavior of the residuals that they generate is limited. Certainly, the results presented in this section indicate that good judgement must be applied by the data analyst to obtain a sensible LAV fit.

REFERENCES

[1] Andrews, D.F., "A Robust Method for Multiple Linear Regression," Technometrics *16*, 523-531 (1974).

[2] Armstrong, R.D., and E.L. Frome, "A Comparison of Two Algorithms for Absolute Deviation Curve Fitting," Journal of the American Statistical Association, *71* (1976), 328-330.

[3] Armstrong, R.D., and J.W. Hultz, "A Restricted Discrete Approximation Problem in the L_1 Norm," SIAM Journal on Numerical Analysis, *14* (1977), 555-565.

[4] Barrodale, I., and F.D.K. Roberts, "An Improved Algorithm for Discrete L_1 Linear Approximation," SIAM Journal on Numerical Analysis *10*, 839-848 (1973).

[5] Barrodale, I., and A. Young, "Algorithms for Best L_1 and L_∞ Linear Approximation on a Discrete Set," Numerical Mathematics *8*, 295-306 (1966).

[6] Beaton, A.E., and Tukey, J.W., "The Fitting of Power Series, Meaning Polynomials, Illustrated on Band-Spectroscopic Data," Technometrics *16*, 147-185 (1974).

[7] Charnes, A., "Optimality and Degeneracy in Linear Programming," Econometrica *20*, 160-170 (1952).

[8] Charnes, A., and W.W. Cooper, "Goal Programming and Constrained Regression — A Comment," Omega *3* (No. 4), 403-409 (1975).

[9] Charnes, A., and W.W. Cooper, "Absolute Deviations and Constrained Regressions," preface to Hilebrand D. translation of De La Valle Poussin, M. Ch. J., "On the Method of Minimum Approxiation," Annales de la Societe de Bruxelles *35*, Part II, 1-16 (1911). In ONR Research Memorandum 96, Carnegie-Mellon University, Pittsburgh, Pennsylvania (1964).

[10] Charnes, A., and W.W. Cooper, *Management Models and Industrial Applications of Linear Programming*, Vols. I and II, (Wiley, New York, 1961).

[11] Charnes, A., W.W. Cooper, and R.J. Niehaus, "Studies in Manpower Planning," U.S. Navy Office of Civilian Manpower Management, Washington, D.C. (1972).

[12] Charnes, A., W.W. Cooper, and R. Ferguson, "Optimal Estimation of Executive Compensation by Linear Programming," Management Science *2*, 138-151 (1955).

[13] Gentleman, J.F., and Wilk, M.B., "Detecting Outliers in a Two-Way Table 1. Statistical Behavior of Residuals," Technometrics *17*, 1-14 (February 1975).

[14] Gentleman, J.F., and M.B. Wilk, "Detecting Outliers, II, Supplementing the Direct Analysis of Residuals," Biometrics *31* 387-410 (1975).

[15] Glover, F., Karney, D., and Klingman, D., "The Augmented Predecessor Index Method for Locating Stepping Stone Paths and Assigning Dual Prices in Distribution Problems," Transportation Science *6*, 171-180 (1972).

[16] Glover, F., Karney, D., Klingman, D., and Napier, A., "A Computational Study on Start Procedures, Basis Change Criteria, and Solution Algorithms for Transportation Problems," Management Science *20*, 793-814 (1975).

[17] Glover, F., D. Klingman, and A. Napier, "An Efficient Dual Approach to Network Problems," Opsearch 9, 1-19 (1972).

[18] Hadley, G., *Linear Programming* (Addison-Wesley, Reading, Massachusetts, 1962).

[19] Hampel, Frank R., "A General Qualitative Definition of Robustness," Annals of Mathematical Statistics *42*, 1887-1896 (1971).

[20] Harter, H.L., "The Method of Least Squares and Some Alternatives — Part V," International Statistical Review *43*, 269-278 (1975).

[21] Hogg, R.V., "Adaptive Robust Procedures; A Partial Review and Some Suggestions For Future Applications and Theory," Journal of the American Statistical Association *69*, (December, 1974).

[22] Hogg, R.V., and Randles, R.H., "Adaption Distribution — Free Regression Methods and Their Application," Technometrics *17*, 399-407 (1975).

[23] Huber, P.J., "Robust Statistics: A Review," The Annals of Mathematical Statistics, *43*, 1041-1067 (1972).

[24] Lee, S.M., *Goal Programming for Decision Analysis* (Auerbach, Philadelphia, 1972).

[25] Mallows, C.L., "Discussion of Invited Papers," Technometrics *16*, 187-188 (1974).

[26] McNeill, J.J., and J.W. Tukey, "Higher-Order Diagnosis of Two-Way Tables, Illustrated on Two Sets of Demographic Empirical Distribtuions," Biometrics *31*, 487-510 (1975).

[27] Prescott, P., "An Approximate Test for Outliers in Linear Models," Technometrics *17*, 129-132 (1975).

[28] Robers, P.D., and A. Ben-Israel, "An Interval Programming Algorithm for Discrete Linear L_1 Approximation Problems," Journal of Approximation Theory *2*, 323-336 (1969).

[29] Searle, S.R., *Linear Models* (Wiley, New York, 1971).

[30] Spyropoulos, K., E. Kiountouzis, and A. Young, "Discrete Approximations in the L_1 Norm," The Computer Journal *16*, 180-186 (1973).

[31] Tukey, J.W., *Exploratory Data Analysis* (Addison-Wesley, Reading, Massachusetts, 1977).

[32] Wagner, H.M., "Linear Programming Techniques for Regression Analysis," Journal of the American Statistical Association *54*, 206-212 (1959).

MULTIPRODUCT LOT-SIZE SCHEDULING WITH PROPORTIONAL PRODUCT DEMANDS

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ABSTRACT

In this paper we consider the multiproduct, multiperiod production-scheduling model of Manne under the assumption that, across products, demands are interrelated over time. When demand requirements are proportional we show that the solution has a specific structure determined by the ratio of setup to production-run time of each product. This structure holds for any length horizon and may permit a substantial (time) savings for column generation solution procedures.

1. PROBLEM DEFINITION

We consider a multiproduct, multiperiod, production-scheduling problem in which demand over a finite T-period horizon is known deterministically. A firm must schedule batch sizes of n products for a T-period horizon. Since the products compete for limited resources, this situation has been termed the capacitated lot-size problem. A linear programming formulation due to Manne [4] has been a classic reference for the capacitated problem. This linear programming model utilizes decision variables which represent a class of dominant sequences. One generates the class of dominant sequences via the observation that if $\{y_t\}$, $t = 1, 2, \ldots, T$, represents the production for a particular product at time t and I_t represents the ending inventory of the product for period t then only sequences for which $y_t \cdot I_{t-1} = 0$ need be considered. For example, if $T = 3$ and the demands for a particular product are 10, 15, and 5 units for periods one, two and three, then one can restrict consideration to $2^{T-1} = 4$ different sequences. These four viable alternatives would be

(30, 0, 0),
(25, 0, 5),
(10, 20, 0),
(10, 15, 5),

where (30, 0, 0) represents a sequence where a setup is incurred in period 1 and 30 units are processed; in periods 2 and 3 no production is scheduled, and demand is satisfied from inventory. We say that period $t \neq T$ is a regeneration point if $I_t = 0$.

The Manne [4] formulation essentially minimizes the amount of setup time required in a finite horizon. Let a_i represent the setup time for product i and b_i the unit processing time. Manne [4] defines the coefficients $\gamma_{ij\tau}$ = hours required for product i in period τ using sequence j. If (1, 0, 1) represents sequence j in a three-period horizon for which setups occur in periods one and three, then

$$\gamma_{ij1} = a_i + b_i\,(r_{i1} + r_{i2}),$$
$$\gamma_{ij2} = 0,$$
$$\gamma_{ij3} = a_i + b_i\,r_{i3},$$

where $r_{i\tau}$ represents the requirements (demand) for product i in period τ. For each product one must choose an appropriate sequence within the constraints of available hours in each period. Let S_τ and V_τ represent respectively the amount of straight time and overtime available in period τ. Define x_{ij} as the proportion of demand for product i that is satisfied via sequence j and l_τ the amount of overtime to be scheduled in period τ. The linear program of Manne [4] is

(Manne)*

$$\min \quad \sum l_\tau,$$

$$(w_i)\ (a) \quad \sum_{j \in J} x_{ij} = 1, \qquad i = 1, 2, \ldots, n,$$

$$(u_\tau)\ (b) \quad \sum \gamma_{ij\tau}\, x_{ij} - l_\tau \leqslant S_\tau, \qquad \tau = 1, 2, \ldots, T,$$

$$(z_\tau)\ (c) \quad l_\tau \leqslant V_\tau, \qquad \tau = 1, 2, \ldots, T,$$

$$x_{ij},\ l_\tau \geqslant 0.$$

We shall associate dual variables $(w_i,\ u_\tau,\ z_\tau)$ along with constraint categories (a), (b), and (c). Observe that the number of structural variables would be $n2^{T-1} + T$ and the number of constraints would be $n + 2T$.

One difficulty in implementing this linear programming formulation relates to its size. For large T and n, several thousand columns may be required. A discussion of these difficulties is given by Kortanek, Sodaro, and Soyster [2]. Certain advanced programming techniques have been formulated so that not all of the alternative sequences need to be explicitly considered. Dzielinski and Gomory [1] and Lasdon and Terjung [3] have developed column generating procedures to cope with the size complexities. Kortanek, Sodaro, and Soyster [2] and May [5] have suggested certain simplifying formulations of the capacitated lot-size problem.

The column-generation procedures are motivated by the relative ease with which the uncapacitated lot-size problem can be solved. In Wagner and Whitin [9] and Wagner [7] a highly efficient dynamic-programming algorithm is developed to handle the uncapacitated problem. A detailed discussion of these techniques is given in Wagner [8].

In this paper we consider the capacitated lot-size problem for which product demands have a special structure. We make the following definition:

DEFINITION: Let $\mathbf{r}_1 = (r_{11}, r_{21}, r_{31}, \ldots, r_{n1})$ be the vector of first period demands for all products. Product demand is said to be *intertemporally proportional* if there exists $\alpha_\tau > 0$,

*We shall take $S_\tau = 0$ so that the objective function is equivalent to minimizing total setup time.

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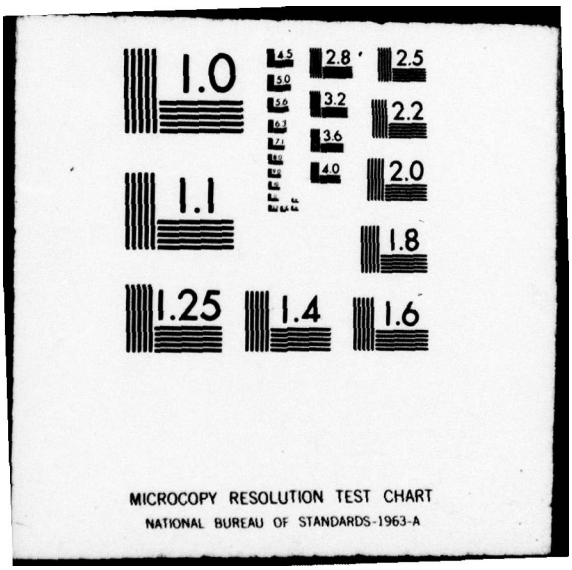
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$\tau = 2, 3, \ldots T$, such that $\mathbf{r}_\tau = \alpha_\tau \mathbf{r}_1$ for $\tau = 2, 3, \ldots T$, where $\mathbf{r}_\tau = (r_{1\tau}, r_{2\tau}, r_{3\tau}, \ldots, r_{n\tau})$ is the vector of requirements in period τ.

This proportionality was observed in [2], where the product line was various polyurethane parts that were fabricated for the interior of automobiles. For example, left and right arm rests come in pairs, and for each pair a dashboard cover was fabricated. One should expect that this is often the case, e.g., the production of chairs and tables is dictated by complementary consumer demands. Products with several components and subassemblies would mandate certain proportional demands, at least in the long run. The main purpose of this paper is to characterize the form of the optimal solution to the capacitated lot-size problem when the product demand remains in a fixed proportion from period to period.

When the demand pattern is intertemporally proportional (or proportional for short) certain surrogate measures of scheduling characteristics are also maintained. Suppose the n products are renumbered so that

$$a_1/(b_1 r_{11}) > a_2/(b_2 r_{21}) > \ldots > a_n/(b_n r_{n1}), \tag{1-1}$$

i.e., product 1 has the largest ratio of setup to first-period processing time. Note that this set of inequalities would remain intact for any subhorizon of demand if the demand pattern is proportional.

2. CAPACITATED LOT-SIZE SOLUTIONS WITH PROPORTIONAL DEMAND

For the capacitated lot-size problem in a T-period horizon, one specifies the demands for each product $i = 1, 2, \ldots, n$ and each period $\tau = 1, 2, \ldots, T$, say $r_{i\tau}$. The demand requirements are conveniently summarized in matrix format (Fig. 1). The assumption that the demand is intertemporally proportional means that each row in the above matrix is a positive multiple of the first row. We assume that the products have been renumbered, if necessary, so that (1-1) holds.

Periods \ Product	1	2	3	...	n
1	r_{11}	r_{21}	r_{31}	...	r_{n1}
2	r_{12}	r_{22}	r_{32}	...	r_{n2}
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
T	r_{1T}	r_{2T}	r_{3T}	...	r_{nT}

FIGURE 1

The main results of this paper concern the form of optimal solutions to (Manne) when the demand is proportional, and they concern computational shortcuts based on the form of the solutions. An optimal solution to (Manne) specifies batch sizes or production levels for each

product in each period of the T-period horizon. Consider batch sizes for each product in period 1. If $r_{i1} > 0$ for each i, then each product must be set up in period 1. In general, the first-period batch sizes specified by an optimal solution to (Manne) can be illustrated as shown in Fig. 2. The shaded areas in Fig. 2 represent the time periods of demand satisifed by batches scheduled in period 1 for the 4-period, 10-product example. In Theorem 2 and Corollary 2.1 it will be shown that when (1-1) holds the schematic diagram will always exhibit a triangular shape (Fig. 3).

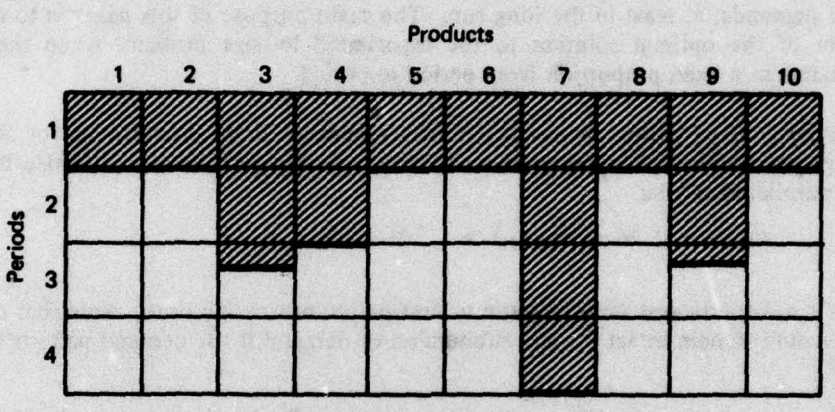

FIGURE 2

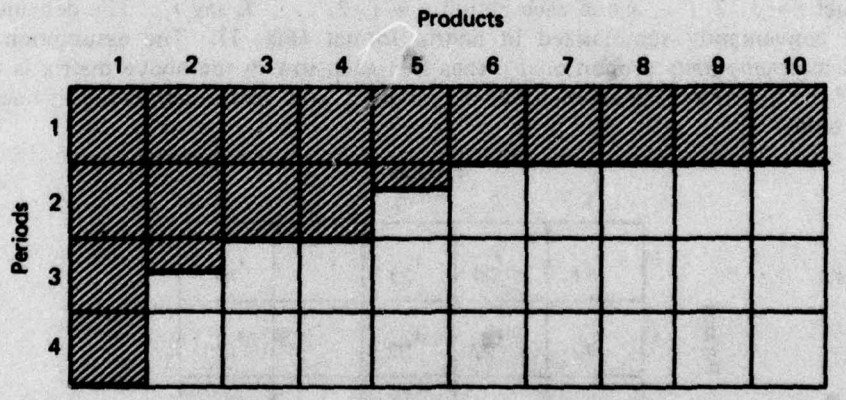

FIGURE 3

DEFINITION: A solution to (Manne) is *triangular* if, when $i_1 \leqslant i_2$ and i_1 and i_2 are being produced in the same period, then the number of periods of demand satisfied by the batch for product i_1 will always be greater than or equal to the number of periods of demand satisfied by the batch for product i_2.

To establish the existence of triangularity we need the following theorems. An important duality result of (Manne) deals with the monotonicity of the optimal dual variables associated with (b), $\{u_t\}$. Manne [4] suggested that the following theorem should be true. We include a proof of this theorem in the appendix.

THEOREM 1: The dual variables $\{u_t\}$ are nonincreasing i.e.,

$$u_\tau^* \geq u_{\tau+1}^*, \ \tau = 1, 2, \ldots, T-1.$$

This is an intuitive result, because anything that is produced in a later time period could be produced earlier if time were available. This makes an earlier time period at least as valuable as a later time period since there is no holding cost for inventory.

Next, we review a recent result [6] concerning the sensitivity of the unconstrained lot-size problem. Consider a horizon length T and let r_τ be the demand for a single product in period τ. Assume that the *cost* for a lot y_τ in period τ is

$$f(y_\tau) = \begin{cases} c_\tau + d_\tau y_\tau & \text{if } y_\tau > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{2-1}$$

where c_τ is the *cost* of a setup in period τ and d_τ is the unit production *cost* in period τ. The dynamic lot-size problem seeks the vector $(y_1, y_2, \ldots, y_T)$ which minimizes

$$\sum_{\tau=1}^{T} f(y_\tau). \tag{2-2}$$

Let I_τ be the ending inventory in period τ; if $\{y_\tau^*\}$, $\{I_\tau^*\}$ is an optimal solution then period $\bar{\tau}$ is called a regeneration point if $I_{\bar{\tau}}^* = 0$ $(\bar{\tau} \neq T)$.

Now consider two alternative setup cost structures,

(i) $c_1 \geq c_2 \geq \ldots \geq c_T$,

(ii) $\alpha c_1 \geq \alpha c_2 \geq \ldots \geq \alpha c_T$,

where $\alpha \in (0, 1)$. In both cases the setup costs are nonincreasing, but in case (ii) the setup costs are proportionately reduced by the same factor in each time period.

Consider an optimal solution for (2-2) when the setup costs are given by (i). An optimal solution is conveniently specified via the set of regeneration points for the given sequence. For example, by $(h_1, h_2, \ldots, h_r)$ we mean that h_1 is the period in which the first regeneration point occurs, h_2 the second, and so on. In this case r regeneration points occur. (Note that the sequence in which a single setup occurs, i.e., the initial period, has no regeneration points, $r = 0$). The following theorem is proven in [6].

THEOREM 2: Assume that $c_\tau \geq c_{\tau+1}$ and $d_\tau \geq d_{\tau+1}$ for $\tau = 1, 2, \ldots, T-1$, and let $(h_1, h_2, \ldots, h_r)$ be an optimal solution to (2-2) for setup costs given by (i). Then for setup costs given by (ii), i.e., proportionally smaller, there exists an optimal solution $(k_1, k_2, \ldots, k_{r'})$ with the following properties: $r' \geq r$ and $k_\tau \leq h_\tau$ for $\tau = 1, 2, \ldots, r$.

The essence of this theorem is simple. If setup costs are reduced then the number of regeneration points (and hence setups) can only increase. Furthermore, since $k_1 \leq h_1$, the first-period batch size will not increase if setup costs are reduced. In general, if $\bar{\tau}$ is a regeneration point for both cost structures, then the lot size scheduled in period $\bar{\tau} + 1$ for the higher setup costs would be at least as large as the lot size in $\bar{\tau} + 1$ for the lower setup costs. This follows since $\bar{\tau}$ is a regeneration point, so that the optimal solution to both cost structures for the horizon $[\bar{\tau} + 1, T]$ is identical to the corresponding periods in the optimal solution for the horizon $[1, T]$.

Next let us return to the consideration of (Manne) when the product demand is intertemporally proportional. Suppose that $(w_i^*, u_\tau^*, z_\tau^*)$ is an optimal dual solution. It then follows from dual feasibility that for each product i

$$\sum_{\tau=1}^{T} \gamma_{ij\tau} u_\tau^* + w_i^* \geq 0, \quad \forall j \in J, \tag{2-3}$$

and, moreover, if $\{x_{ij}^*\}$ is an optimal primal solution and $x_{ij_i}^* > 0$, then (2-3) is a strict equality for $j = j_i$. In fact, these duality considerations are often used to sequentially generate candidate sequences when the set J is very large [1]. The optimal sequence j (or set of sequences) for product i can be obtained by minimizing the left-hand side of (2-3). For a given product i, w_i^* is a constant, and an optimal sequence is one which minimizes (over $j \in J$) the quantity

$$\sum_{\tau=1}^{T} \gamma_{ij\tau} u_\tau^*. \tag{2-4}$$

The general form of (2-4) for some arbitrary sequency j with regeneration points $(h_1, h_2, \ldots, h_r)$ is

$$u_1^* \left[a_i + b_i \sum_{\tau=1}^{h_1} r_{i\tau}\right] + u_{h_1+1}^* \left[a_i + b_i \sum_{\tau=h_1+1}^{h_2} r_{i\tau}\right] + \ldots + u_{h_r+1}^* \left[a_i + b_i \sum_{\tau=h_r+1}^{T} r_{i\tau}\right]. \tag{2-5}$$

For the case in which product demand is proportional, i.e., $r_{i\tau} = \alpha_\tau r_{i1}$, (2-5) can be written as

$$u_1^* \left[a_i + b_i r_i \sum_{\tau=1}^{h_1} \alpha_\tau\right] + u_{h_1+1}^* \left[a_i + b_i r_i \sum_{\tau=h_1+1}^{h_2} \alpha_\tau\right] + \ldots + u_{h_r+1}^* \left[a_i + b_i r_i \sum_{\tau=h_r+1}^{T} \alpha_\tau\right], \tag{2-6}$$

or equivalently,

$$b_i r_i \left[\left(c_{i1} + d_1 \sum_{\tau=1}^{h_1} \alpha_\tau\right) + \left(c_{i,h_1+1} + d_{h_1+1} \sum_{\tau=h_1+1}^{h_2} \alpha_\tau\right) + \ldots + \left(c_{i,h_r+1} + d_{h_r+1} \sum_{\tau=h_r+1}^{T} \alpha_\tau\right)\right], \tag{2-7}$$

where $c_{i\tau} = \dfrac{u_\tau^* a_i}{b_i r_i}$ and $d_\tau = u_\tau^*$. Observe, then, that $c_{i\tau} \geq c_{i,\tau+1}$ and $d_\tau \geq d_{\tau+1}$, since $u_\tau^* \geq u_{\tau+1}^*$. Since (2-7) is the general form of (2-4) for each product i and sequence j, the following corollary is obtained:

COROLLARY 2.1 (Triangularity): Assume that the products are numbered so that $a_1/b_1 r_{11} > a_2/b_2 r_{21} > \ldots > a_n/b_n r_{n1}$ and that the demand is intertemporally proportional throughout the T-period horizon. There exists an optimal basic solution $\{x_{ij}^*\}$ to (Manne) with

the following property. Let $i_1 < i_2$, and suppose that $x^*_{i_1 j_1}$ is basic. If products i_1 and i_2 are produced in period p and product i_1 has its first regeneration point after p in period p_1, then, for any sequence j_2 for which $x^*_{i_2 j_2}$ is basic, a first regeneration after p exists at p_1 or earlier.

PROOF: The proof follows directly from Theorem 2 and (2-7). Observe that if $i_1 \leqslant i_2$ then

$$c_{i_2\tau} = \frac{(a_{i_2}/b_{i_2}\, r_{i_2})}{(a_{i_1}/b_{i_1}\, r_{i_1})}\; c_{i_1\tau} \leqslant c_{i_1\tau} \quad \text{for each } \tau.$$

and d_τ is the same for both products. Since $\{c_{i_1\tau}\}$, $\{c_{i_2\tau}\}$, and $\{d_\tau\}$ are nonincreasing in τ, the corollary is a consequence of Theorem 2 and of the fact that minimization of (2-7) yields optimal sequences j_1 and j_2 for products i_1 and i_2. QED.

The corollary characterizes the form of optimal solutions to (Manne) when the product demand is proportional. In particular, the first-period batch sizes vary in accordance with the ratio of setup to run time. A product with a large ratio of setup to run time should have a first-period batch size that is greater than or equal to the batch size for a product with a smaller ratio of setup to run time. This corollary also corrects an improper assertion made in [2]. In [2] it was erroneously asserted that, when demand is proportional, before any demand in excess of $r_{i_2 1}$ is scheduled in period 1 for product i_2, all demand for product i_1 would be scheduled in period 1, $i_1 < i_2$.

COROLLARY 2.2 (Monotone Set-ups): Assume that the products are numbered so that $a_1/b_1 r_{11} > a_2/b_2 r_{21} > \ldots a_n/b_n r_{n1}$ and that the demand is intertemporally proportional throughout the T-period horizon. There exits an optimal basic solution $\{x^*_{ij}\}$ to (Manne) with the following property. Let $i_1 < i_2$ and suppose that $x_{i_1 j_1}$ is basic. There exists a sequence j_2 with at least as many setups as j_1 for which $x_{i_2 j_2}$ is basic.

PROOF: The result follows directly from Theorem 2 in a manner analogous to the proof of Corollary 2.1.

3. EXAMPLES AND USES OF TRIANGULARITY

For the case in which demand is intertemporally proportional, the optimal basic solutions to (Manne) exhibit a regular pattern as illustrated by Fig. 3. An algebraic interpretation exists in terms of the set of alternative production sequences. This is illustrated for a four-period horizon. For a horizon of $T = 4$ there exist eight alternative sequences, which we number as follows:

1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1
0	0	0	0	1	1	1	1
0	0	1	1	0	0	1	1
0	1	0	1	0	1	0	1

Observe that the sequences are numbered in order of decreasing first regeneration point, if a tie exists then according to decreasing order or second regeneration point, and so on. An implication of Corollary 2.1 is that if $x^*_{i_1 j}$ is basic for $j = 5, 6, 7,$ or 8, and $i_1 < i_2$, then $x^*_{i_2 j}$ is not basic for $j = 1, 2, 3,$ or 4.

TABLE 1

Products	a_i (hr)	b_i (hr)	Demand 1	2	3	4
1	9	0.1	21	35	56	42
2	4	0.2	6	10	16	12
3	8	0.4	12	20	32	24
4	4	0.5	6	10	16	12
5	8	0.7	12	20	32	24
6	8	0.8	15	25	40	30
7	3	0.6	9	15	24	18
8	3	0.5	27	45	72	54
9	1	0.5	12	20	32	24
10	1	0.7	24	40	32	48

Actually, a much stronger result is implied by Corollary 2.1. If $i_1 < i_2$ and $x^*_{i_1 j_1}$ is basic, then for any basic $x^*_{i_2 j_2}$ it follows, according to the given numbering system, that $j_2 \geq j_1$. Consider the example with ten products in Table 1. For resource availabilities let $V_1 = 200$, $V_2 = 150$, $V_3 = 150$, $V_4 = 150$, and all $S_\tau = 0$. The (unique) optimal basic solution to (Manne) produces the following sequences:

Product	Sequence
1	1
2	1
3	1
4	1
5	1,3
6	3,5
7	5,6
8	6
9	6
10	8

Corollary 2.2 further constrains the allowable occurrences of the alternative sequences for proportional demand. Note that sequence 4 has a first regeneration point (period 2) later than sequence 5 (period 1), but sequence 4 contains three setups and sequence 5 only two setups. If $i_1 < i_2$ and $x^*_{i_1 4}$ is basic, then an optimal basic solution with $x^*_{i_2 j} > 0$ exists for $j = 6$, 7, or 8. Taken together, Corollaries 2.1 and 2.2 imply that in a given program an optimal solution exists such that sequences 4 and 5 may not occur with different products. At most they can occur simultaneously for one single product. For example, if $i_1 < i_2$, then $x^*_{i_1 4}$ and $x^*_{i_2 5}$ both basic violates Corollary 2.2, and $x^*_{i_1 5}$ and $x^*_{i_2 4}$ both basic violates Corollary 2.1.

One of the most efficient approaches to solving (Manne) is to combine column generation with generalized upper bounding. That is, for each product use the dual variables as an opportunity cost for time, $u_\tau a_i$ for the setup cost for product i in period τ, and $u_\tau b_i$ for the production cost per unit in period τ. Then for each product solve the Wagner-Whitin problem to

find the lowest-cost production sequence. This finds the production sequence for each product with the lowest reduced cost in the linear programming sense (see Dzielinski and Gomory [1]). After the Wagner-Whitin problem is solved for each product, the sequence with the lowest reduced cost is pivoted into the basis. The column-generation linear programming iterations are continued until the products with the lowest reduced costs are already in the basis or would not improve the solution when pivoted into the basis, i.e., a reduced cost of zero. Now suppose that this column-generation approach is applied to the numerical example of this section and that the foregoing optimal solution is obtained at some iteration. For illustrative purposes suppose that this feasible solution is not optimal and subsequent iterations are required. How can triangularity simplify the column-generation procedure? If at this current iteration it is true that $\{u_\tau\}$ are nonincreasing, then according to Corollaries 2.1 and 2.2 only a small number of products and sequences need to be priced for possible entry into the next linear programming solution. In particular, Corollaries 2.1 and 2.2 state that those sequences which maintain triangularity will always provide a lower reduced cost. For the example problem, one should only price sequence 2 for product 5 and sequence 7 for products 9 and 10. One would not need to price sequences for the other products. The product-sequence combination with the lowest reduced cost would be among this set of three alternatives. The column-generation routine could be structured to eliminate the generation of all sequences for a large number of products and to reduce significantly the number of alternative sequences for the remaining products.

The impact of this result upon column-generation procedures for (Manne) has not been tested; increased efficiency will depend upon the form of intermediate solutions, i.e., at what point intermediate solutions provide monotone nonincreasing dual variables and triangular-type feasible solutions. However, since at optimality the dual variables must be monotone nonincreasing, adding constraints to the dual of (Manne), i.e., columns to the primal, which enforce monotonicity does not affect the optimal solution. The form of these constraints in the dual would be simply $u_\tau \geqslant u_{\tau+1}$ for $\tau = 1, 2, \ldots, T-1$. In this manner, the monotonicity requirement is maintained at each iteration. The trade-off here is that if one or more of these $T-1$ artificial columns is positive at some intermediate iteration then the current solution is not feasible for (Manne). The overall efficiency gained by employing these triangularity results will be explored in a subsequent paper.

4. CONCLUSION

Given proportional product demands, we have shown that a triangularity property exists which greatly simplifies the computational needs for solving Manne's production-scheduling model. These results are useful under certain circumstances when product proportionality is not met. First, in the illustrations we have assumed that all products had requirements to be met in every period. In reality there is inventory to meet early requirements for many products. Therefore, the requirements to be met from production are not proportional. But Corollary 2.1 shows that triangularity holds from the point of first production onward. The triangular structure for period τ production is for products with requirements in period τ not met by inventory.

Second, since demand proportionality is not a requirement for monotonically nonincreasing dual variables, if there is a subset of products with demand proportionality within a larger set of products, triangularity holds for the subset. There may be many subsets having demand proportionality within the subsets. Again, triangularity and the column-generation reduction procedures hold within the subsets.

REFERENCES

[1] Dzielinski, B. P., and R. E. Gomory, "Optimal Programming of Lot Sizes, Inventory and Labor Allocations," Management Science *11*, 874-890 (1965).

[2] Kortanek, K. O., D. Sodaro, and A. L. Soyster, "Multi-Product Production Scheduling via Extreme Point Properties of Linear Programming," Naval Research Logistics Quarterly *15*, 287-300 (1968).

[3] Lasdon, L. S., and R. C. Terjung, "An Efficient Algorithm for Multi-Item Scheduling," Operations Research *19*, 946-969 (1971).

[4] Manne, A. S., "Programming of Economic Lot Sizes," Management Science *4*, 115-135 (1958).

[5] May, J. G., "A Linear Program for Economic Lot Sizes Using Labor Priorities," Management Science *21*, 277-285 (1974).

[6] Murphy, F. H., and A. L. Soyster, "Sensitivity Analysis of the Cost Parameters in the Dynamic Lot Size Model," Working paper, Virginia Polytechnic Institute and State University, Blacksburg, Virginia (1977).

[7] Wagner, H. M., "A Postscript to Dynamic Problems in the Theory of the Firm," Naval Research Logistics Quarterly *7*, 7-12 (1960).

[8] Wagner, H. M., Principles of Operations Research (Prentice-Hall, Englewood Cliffs, N.J., 1969).

[9] Wagner, H. M., and T. M. Whitin, "A Dynamic Version of the Economic Lot Size Model," Management Science *5*, 89-96 (1958).

APPENDIX

THEOREM 1: The dual variables $\{u_\tau^*\}$ are nonincreasing, i.e.,

$$u_\tau^* \geq u_{\tau+1}^*, \qquad \tau = 1, 2, \ldots, T-1.$$

PROOF: We first show that $u_1^* \geq u_2^*$ and then use an induction step to complete the proof.

To show that $u_1^* \geq u_2^*$, first note that if no production is scheduled in period 2 then $u_2^* = 0$. In this case obviously $u_1^* \geq u_2^*$. Next, suppose that production for some product i is scheduled in period 2, i.e., $x_{ij_1}^* > 0$ for some sequence j_1. Consider the sequence j_2 that differs from j_1 in that no setup is specified for period 2. Using complementary slackness and dual feasibility of problem (Manne) one obtains

$$\text{(A1)} \qquad \sum_{\tau=1}^{T} \gamma_{ij_1\tau}\, u_\tau^* + w_i^* = 0$$

and

$$\text{(A2)} \qquad \sum_{\tau=1}^{T} \gamma_{ij_2\tau}\, u_\tau^* + w_i^* \geq 0.$$

Subtracting (A1) from (A2), we obtain

$$\text{(A3)} \qquad \sum_{\tau=1}^{T} (\gamma_{ij_2\tau} - \gamma_{ij_1\tau})\, u_\tau^* \geq 0,$$

and since, $\gamma_{ij_2\tau} = \gamma_{ij_1\tau}$ for $\tau = 3, 4, \ldots, T$, it follows from (A3) that

$$\text{(A4)} \qquad \sum_{\tau=1}^{2} (\gamma_{ij_2\tau} - \gamma_{ij_1\tau})\, u_\tau^* \geq 0.$$

Now let τ', $2 \leqslant \tau' \leqslant T$, be the latest-period demand that is scheduled in period 2 according to sequence j_1 (which is also the latest-period demand scheduled in period 1 for sequence j_1). From (A4) it follows that

$$\text{(A5)} \qquad [(a_i + b_i \sum_{\tau=1}^{\tau'} r_{i\tau}) - (a_i + b_i r_{i1})]\, u_1^* - (a_i + b_i \sum_{\tau=1}^{\tau'} r_{i\tau})\, u_2^* \geqslant 0,$$

which, after simplification, yields

$$\text{(A6)} \qquad (u_1^* - u_2^*)\, b_i \sum_{\tau=2}^{\tau'} r_{i\tau} \geqslant u_2^*\, a_i.$$

Since $a_i \geqslant 0$, then (A6) implies $u_1^* \geqslant u_2^*$.

Now suppose that $u_1^* \geqslant u_2^* \geqslant \ldots \geqslant u_\tau^*$. We will show that $u_\tau^* \geqslant u_{\tau+1}^*$. Again, if no production is scheduled in period $\tau + 1$, then $u_{\tau+1}^* = 0$ and the conclusion is clear. Hence, suppose that production for some product i is scheduled in period $\tau + 1$, i.e., $x_{ij_1} >$ for some sequence j_1. We need to consider two cases:

(1) Sequence j_1 specifies a setup in period τ.

(2) Sequence j_1 specifies no setup in period τ.

For case (1) consider the sequence j_2 that differs from j_1 only in that there is no setup in period $\tau + 1$. As before, it follows from complementary slackness and dual feasibility that

$$\text{(A8)} \qquad \sum_{\tau=1}^{T} (\gamma_{ij_2\tau} - \gamma_{ij_1\tau})\, u_\tau^* \geqslant 0.$$

Since sequences j_1 and j_2 differ only in periods τ and $\tau + 1$, (A8) implies

$$\text{(A9)} \qquad (\gamma_{ij_2\tau} - \gamma_{ij_1\tau})\, u_\tau^* + (\gamma_{ij_2,\tau+1} - \gamma_{ij_1,\tau+1})\, u_\tau^* \geqslant 0.$$

If τ', $\tau + 1 \leqslant \tau' \leqslant T$, is the latest-period demand scheduled in period $\tau + 1$ for sequence j_1 (which is also the latest-period demand scheduled in period τ for sequence j_2), then from (A9) one obtains

$$\text{(A10)} \qquad (b_i \sum_{t=\tau+1}^{\tau'} r_{it})\, u_\tau^* - (a_i + b_i \sum_{t=\tau+1}^{\tau'} r_{it})\, u_{\tau+1}^* \geqslant 0.$$

Rearranging (A10) we obtain

$$\text{(A11)} \qquad (u_\tau^* - u_{\tau+1}^*)\, b_i \sum_{t=\tau+1}^{\tau'} r_{it} \geqslant u_{\tau+1}^*\, a_i,$$

which implies that $u_\tau^* \geqslant u_{\tau+1}^*$.

For case (2) let sequence j_2 be defined as in case (1), i.e., sequence j_2 has a setup in period τ, but no setup in period $\tau + 1$. Let $\hat{\tau} < \tau$ be the period for which period τ demand is scheduled according to sequence j_1. Again, it follows that

$$\text{(A12)} \qquad \sum_{\tau=1}^{T} (\gamma_{ij_2\tau} - \gamma_{ij_1\tau})\, u_\tau^* \geqslant 0,$$

which simplifies to

(A13) $$(\gamma_{ij_2\tau} - \gamma_{ij_1\tau})\, u^*_\tau + (\gamma_{ij_2\tau} - \gamma_{ij_1\tau})\, u^*_\tau + (\gamma_{ij_2,\tau+1} - \gamma_{ij_1,\tau+1})\, u^*_{\tau+1} \geqslant 0.$$

Substitution for the three terms in (A13) yields

(A14) $$-b_i\, r_{i\tau}\, u^*_\tau + (a_i + b_i \sum_{t=\tau}^{\tau'} r_{it})\, u^*_\tau + (-a_i - b_i \sum_{t=\tau+1}^{\tau'} r_{it})\, u^*_{\tau+1} \geqslant 0,$$

where τ' remains as previously defined. Next we use the induction hypothesis, $u^*_\tau \geqslant u^*_\tau$, in (A14), which yields

(A15) $$(a_i + b_i \sum_{t=\tau+1}^{\tau'} r_{it})\, u^*_\tau \geqslant (a_i + b_i \sum_{t=\tau+1}^{\tau'} r_{it})\, u^*_{\tau+1}$$

Now case (2) follows from (A15), and the theorem is proven.

AN EXACT BRANCH-AND-BOUND PROCEDURE FOR THE QUADRATIC-ASSIGNMENT PROBLEM

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ABSTRACT

The quadratic-assignment problem is a difficult combinatorial problem which still remains unsolved. In this study, an exact branch-and-bound procedure, which is able to produce optimal solutions for problems with twelve facilities or less, is developed. The method incorporates the concept of stepped fathoming to reduce the effort expended in searching the decision trees. Computational experience with the procedure is presented.

1. INTRODUCTION

The quadratic-assignment problem is a combinatorial problem that has been of great interest to many researchers. The problem can be stated as follows:

$$\text{Minimize} \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m} c_{ijkl}\, x_{ij}\, x_{kl} + \sum_{i=1}^{m}\sum_{j=1}^{m} f_{ij}\, x_{ij}$$

$$\text{subject to} \sum_{j=1}^{m} x_{ij} = 1, \qquad i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} x_{ij} = 1, \qquad j = 1, \ldots, m,$$

$$x_{ij} \text{ is 0 or 1}, \qquad i,j = 1, \ldots, m.$$

The problem can be interpreted as follows. We suppose that m facilities or objects are to be assigned to m locations. Here, x_{ij} is 1 if facility i is placed in location j and is 0 otherwise. The quantity c_{ijkl} is the cost of the mutual assigment of object i to location j and object k to location l, and it is usually determined as the number of interactions u_{ik} between objects i and k weighted by the distance from location j to location l, that is, $c_{ijkl} = u_{ik}d_{jl}$. Furthermore, f_{ij} is the fixed cost of assigning facility i to location j.

*This author's work was supported by NSF grant #GK-38337.
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Since Koopmans and Beckman [10] introduced the quadratic-assignment problem in the context of locating indivisible objects, the problem has gained a great deal of popularity among researchers, due mostly to its wide range of applications. In [16], Whitehead and Elders discussed the use of the problem in the area of building layout. In [4], Elshafei described a quadratic-assignment algorithm that can be used in the context of hospital layout. The problem has also been used in the fields of urban planning, control-panel layout, and wiring design in the placement of electronic components in an assembly. For details on these applications, the reader may refer to Hopkins [9], Dorris [3], Breuer [2], Gaschutz and Ahrens [5], and Steinberg [15].

Various procedures for the solution of the problem have been suggested in the literature, including both exact and heuristic procedures. This study concerns itself with exact methods. Currently, the available exact procedures are all of the branch-and-bound type and can be classified into single-assignment algorithms, pair-assignment algorithms, and pair-exclusion algorithms. Single-assignment algorithms proceed by the assignment of one unassigned facility to a vacant location at any stage of the search process. The procedures of Gilmore [7], Graves and Whinston [8], and Lawler [12] fall in this class. Neither Gilmore nor Lawler reported any computational experience. Graves and Whinston compared their procedure with some existing heuristics and provided better-quality solutions, but they did not guarantee optimality. Pair-assignment methods proceed by simultaneous location of two facilities at two unoccupied locations. In [11], Land described a pair-assignment algorithm that first reduces the cost matrix so that it contains a zero in each row and a zero in each column. Gavett and Plyter [6] extended the method of Land by tightening the computation of the lower bounds. They reported that their algorithm took 14 min on an IBM 7044 machine to solve a problem of size $m = 7$ and 42 min for $m = 8$. Pair-exclusion algorithms proceed on the basis of a stage-by-stage exclusion of assignments from a solution to the problem. In [14], Pierce and Crowston reported the results for this procedure for a problem of size four facilities.

In this study, we discuss an exact branch-and-bound scheme for solving the quadratic-assignment problem. The method is similar to that suggested by Gilmore [7], but it differs in the computation of the lower bounds and in the branching rules. It also incorporates the concept of stepped fathoming given in [1] for speeding the search of the decision tree. The reported algorithm was able to find the optimal solution of a problem of size 12 facilities but failed to produce exact solutions for problems of size $m \geqslant 15$.

2. AN EXACT-SOLUTION PROCEDURE

In this section we describe a branch-and-bound solution procedure which can be used to obtain optimal and qualified suboptimal solutions. The following notation will be used. The location to which object i is assigned is denoted by i^*. Hence, the mutual cost of assigning objects i and j is $u_{ij}d_{i^*j^*} + u_{ji}d_{j^*i^*}$ and the fixed cost of these assignments is $f_{ii^*} + f_{jj^*}$.

Decision Tree and General Framework

At each stage of the algorithm, we have a set of objects that has already been assigned to certain locations. These already assigned objects form a *partial solution* of the assignment problem. In order to obtain a feasible solution, that is, a complete assignment, we must find a *completion* of the partial solution. Rather than considering all the possible ways of completing the partial solution, we first investigate whether the partial solution might lead to a complete solution with an objective value smaller than the best solution that we already have. This is done by calculating a *lower bound* on the cost of completing this partial solution.

Let B be the lower bound and let C^* be the cost of the best available assignment. If $B \geqslant C^*$ then any completion of the partial solution can lead to no improvement. In this case, the partial solution is said to be *fathomed*, and it is abandoned. On the other hand, if $B < C^*$ it is worthwhile for us to pursue the partial solution by seeking to assign more objects.

Calculation of the Lower Bound

Suppose that a set of objects indexed by the set I has already been assigned to a set of locations indexed by the set J. In particular, suppose that object i is assigned to location i^*. A lower bound B on the cost of this partial solution and its completion is computed as $B = C_1 + C_2 + C_3$, where

C_1 = cost of the partial assignment;

C_2 = lower bound on the cost of interaction between assigned objects and unassigned objects plus the fixed cost of locating the unassigned objects;

C_3 = lower bound on the cost of interaction among the unassigned objects themselves.

Note that C_1 is given by

$$C_1 = \sum_{i \in I} f_{ii^*} + \sum_{i,j \in I} u_{ij}\, d_{i^*j^*}.$$

Here C_2 is the optimal cost of the following linear-assignment problem:

$$\text{Minimize} \sum_{i \notin I} \sum_{j \notin J} b_{ij} x_{ij}$$

$$\text{subject to} \sum_{j \notin J} x_{ij} = 1 \qquad \text{for } i \notin I,$$

$$\sum_{i \notin I} x_{ij} = 1 \qquad \text{for } j \notin J,$$

$$x_{ij} \geqslant 0 \qquad \text{for } i \notin I,\ j \notin J,$$

where b_{ij} is a bound on the cost resulting from the assignment of object i to location j. For example, we can use

$$b_{ij} = f_{ij} + \sum_{t \in I} (u_{it}\, d_{jt^*} + u_{ti}\, d_{t^*j}).$$

Of course, a complete solution of the linear-assignment problem can be replaced by the simpler task of reduction of the cost matrix (b_{ij}) such that it has at least one zero in each row and each column by subtraction of the minima of the rows from the rows, and the minima of the columns from the resultant columns.

Two methods of computation of C_3 are available. The first method relies on the ranking of the interactions and distances as follows. Rank the interactions u_{ij} in a descending order for $i, j \notin I$, and rank the distances d_{ij} in an ascending order for $i, j \notin J$. This results in an ordered interaction vector and an ordered distance vector. Then C_3 is the inner product of these two vectors. In other words, we calculated C_3 by matching the largest interaction among unassigned elements to the smallest distance between unassigned locations, the second largest interaction to the second smallest distance, and so forth. Clearly, this procedure will give a lower bound on the cost among unassigned elements.

An alternative method for finding a suitable bound C_3 is the solution of a linear assignment problem whose cost matrix is constructed as follows. For each unlocated element i, rank

the interactions between it and all other unlocated elements in descending order. Similarly, for each vacant location j, rank the distances between it and all other vacant locations in ascending order. Then a lower bound e_{ij} on the cost of locating facility i in location j is the inner product of the above two vectors. Thus, we find C_3 by solving the following linear-assignment problem:

$$\text{Minimize} \sum_{i \notin I} \sum_{j \notin J} e_{ij} x_{ij}$$

$$\text{subject to} \sum_{j \notin J} x_{ij} = 1 \qquad \text{for } i \notin I,$$

$$\sum_{i \notin I} x_{ij} = 1 \qquad \text{for } j \notin J,$$

$$x_{ij} \geqslant 0 \qquad \text{for } i \notin I,\ j \notin J.$$

Needless to say, the above assignment problem can be combined with the assignment problem in the C_2 calculation to give $C_2 + C_3$. The overall lower bound $B = C_1 + C_2 + C_3$ is now available.

Continuation of the Search: Fathoming (Backward Move)

Suppose that k objects indexed by the set I have already been assigned to k locations indexed by the set J. The *level* of the search tree is called k. A bound on the cost that results from all completions of the current partial solution B is calculated as discussed above. If $B \geqslant C^*$, where C^* is the best known cost of a complete assignment, then the partial solution is *fathomed*. The last assignment, that is, the kth assignment, is *banned* or *prohibited* in the hope that this will lead to an improved completion. For example, if the kth assignment involves placing object i_k in location i_k^* then $x_{i_k i_k^*}$ is forced to be zero. Here, i_k is placed in the list of unassigned objects, that is, i_k is removed from I, and similarly i_k^* is removed from the list of unassigned locations, that is, i_k^* is removed from J. We calculate a new bound B' in exactly the same manner as explained above, except, of course, that the assignment $x_{i_k i_k^*} = 1$ is prohibited, by forcing $b_{i_k i_k^*} = \infty$ while we solve the linear-assignment problem. If B' is still $\geqslant C^*$, then the partial solution of the first $k-1$ assignments, while banning the assignment i_k to i_k^*, can still lead to no improved solutions. Since the first $k-1$ assignments with $x_{i_k i_k^*} = 1$ and $x_{i_k i_k^*} = 0$ lead to no improvement, then all the possibilities at level k have been exhausted, and prohibition of the assignment at level $k-1$ is now possible. This condition is called *strong fathoming*. The level of the tree is thus reduced by one unit, and the assignment at level $k-1$ is prohibited. If, on the other hand, the bound B' is less than C^*, a condition referred to as *weak fathoming*, then object i_k is assigned to some other unassigned location. This is discussed in more detail in the forward move of the search. The cases of strong and weak fathoming are depicted in Figures 1 and 2.

Progress of the Search (Forward Move)

If $B < C^*$ then we must choose an object i_{k+1} for assignment. For example, we may choose i_{k+1} to be an unassigned object with maximum interactions with already assigned objects, or choose i_{k+1} to be an unassigned object with maximum interactions with the most recently assigned object i_k. This object is assigned to an unassigned location i_{k+1}^* which is not prohibited. This location i_{k+1}^* can be chosen in such a way that the total weighted interaction with assigned objects in minimal. For example, choose i_{k+1}^* which minimizes $\sum_{j=1}^{k} u_{i_{k+1} i_j} d_{ii_j^*} + \sum_{j=1}^{k} u_{i_{k+1} i_j} d_{i_j^* t}$ over $t \notin J$ such that $x_{i_{k+1} t} = 1$ is not prohibited.

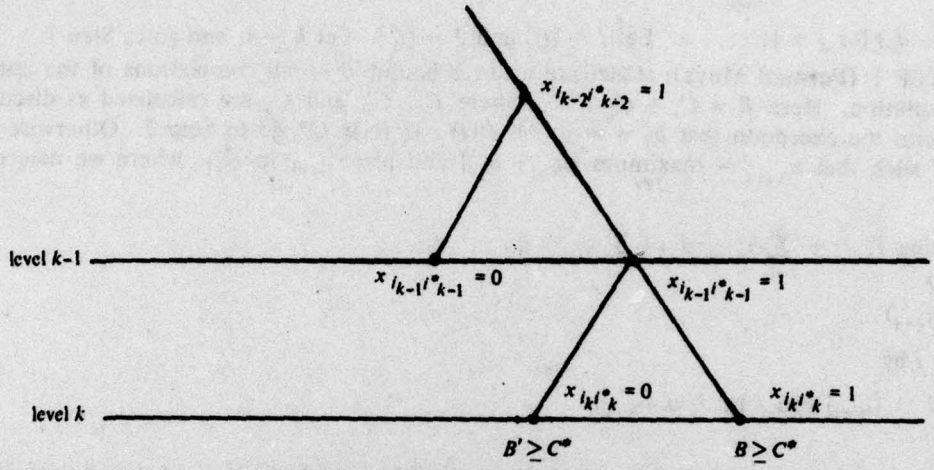

FIGURE 1. Illustration of strong fathoming.

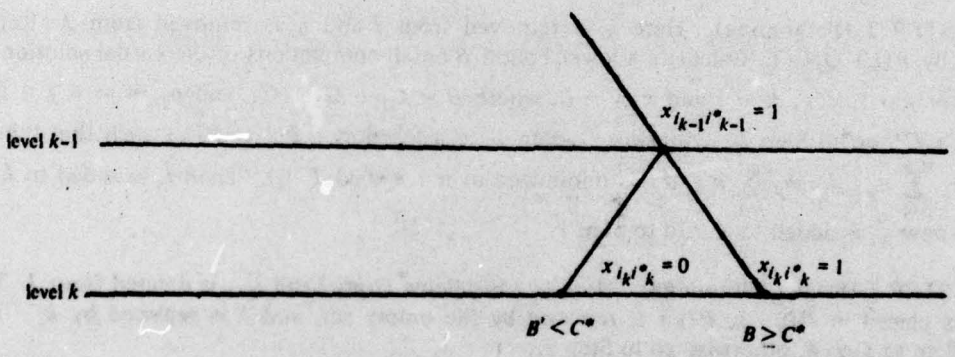

FIGURE 2. Illustration of weak fathoming.

Needless to say, when the level of the tree is m, if the cost is less than C^*, then C^* is updated and the corresponding assignment is stored.

Termination

We have described forward and backward progress of the search tree. If the level of the tree ever reaches value zero, then we stop. This would mean that we are currently at level one, and are trying to backtrack. This means that all possible assignments under $x_{i_1 i_1^*} = 1$ and $x_{i_1 i_1^*} = 0$ have already been enumerated so that all possible ways of assignment of the m objects are enumerated, and we stop. The stored assignment and corresponding C^* give the optimal solution.

Summary of the Algorithm

We have discussed above all the details required to describe the following solution procedure of the quadratic assignment problem:

INITIALIZATION STEP: Let the prohibited locations for object i be $P(i) = \emptyset$ for $i = 1, \ldots, m$, and let $C^* = \infty$. Choose an object i_1 and place it in location i_1^*. We may determine i_1 by maximizing $\sum_{j=1}^{m} (u_{ij} + u_{ji})$ for $i = 1, \ldots, m$, and we can determine i_1^* by minimizing $\sum_{i=1}^{n} (d_{ij} + d_{ji})$ for $j = 1, \ldots, n$. Let $I = \{i_1\}$ and $J = \{i_1^*\}$. Let $k = 1$, and go to Step 1.

STEP 1 (Forward Move): Calculate a lower bound B on all completions of the current partial solution. Here $B = C_1 + C_2 + C_3$, where C_1, C_2, and C_3 are calculated as discussed above with the exception that $b_{ij} = \infty$ if $j \in P(i)$. If $B \geq C^*$, go to Step 2. Otherwise pick $i_{k+1} \notin I$ such that $u_{i_{k+1}i_k} = \underset{i \notin I}{\text{maximum}} [u_{ii_k} + u_{i_k i}]$ and place i_{k+1} in i_{k+1}^*, where we determine i_{k+1}^* by

minimizing $f_{i_{k+1}j} + \sum_{t=1}^{k} u_{i_{k+1}i_t} d_{ji_t^*} + \sum_{t=1}^{k} u_{i_t i_{k+1}} d_{i_t^* j}$

$j \notin J$

$j \notin P(i_{k+1})$

Replace I by

$$I \cup \{i_{k+1}\} \text{ and } J \text{ by } J \cup \{i_{k+1}^*\}.$$

If $k = m - 1$, then C_1 is the cost of the complete assignment. If $C_1 < C^*$, replace C^* by C_1, store (i_t, i_t^*) for $t = 1, \ldots, m$, replace k by m, and go to Step 2. If $C_1 \geq C^*$, then replace k by m and go to Step 2. If $k < m - 1$, then replace k by $k + 1$ and repeat Step 1.

STEP 2 (Fathoming): Here i_k is removed from I and i_k^* is removed from J. Replace $P(i_k)$ by $P(i_k) \cup \{i_k^*\}$. Calculate a lower bound B on all completions of the partial solution $x_{i_t i_t^*} = 1$ for $t = 1, \ldots, k - 1$ and $x_{i_k i_k^*} = 0$, where $B = C_1 + C_2 + C_3$, and $b_{ij} = \infty$ if $j \in P(i)$. If $B \geq C^*$, go to Step 3. Otherwise, assign i_k to a location $t \notin J \cup P(i_k)$ such that the cost $f_{ki} + \sum_{j=1}^{k-1} u_{kj} d_{i_j^* t} + \sum_{j=1}^{k-1} u_{ji_k} d_{i_j^* t}$ is minimized over $t \notin J \cup P(i_k)$. Then i_k is added to I, and t, the new i_k^*, is added to J. Go to Step 1.

STEP 3 (Strong Fathoming): Here i_{k-1} is deleted from I and i_{k-1}^* is deleted from J. Then i_{k-1}^* is placed in $P(i_{k-1})$, $P(i_k)$ is replaced by the empty set, and k is replaced by $k - 1$. If $k = 0$ go to Step 4, otherwise go to Step 2.

STEP 4 (Termination): The search of the decision tree has been completed. The optimal cost is C^*, and its corresponding assignment (i_k, i_k^*), $k = 1, \ldots, m$, is the optimal assignment. Stop.

Note that during the initialization step an upper bound $C^* = \infty$ is used. As the search progresses, C^* denotes the objective value of the best available complete assignment. Further, $P(i)$ represents the locations that object i cannot be assigned to. These are initialized by the empty sets. Step 1 represents a forward step, where the level of the tree increases by one unit. In this case the bound is less than C^*, hence a complete solution with an objective better than C^* is possible. Step 2 is a fathoming step, where $B \geq C^*$. In this case the last assignment is prohibited. Immediately, a new bound is calculated. If the new bound is less than C^*, then a forward move is made. But if the bound is still greater than or equal to C^*, then a strong fathoming is made at Step 3, and the level of the tree is reduced. Of course, strong fathoming is most desirable, since it avoids the expensive task of trying to locate object i_k in a free location other than i_k^*.

Adding New Facilities to an Existing Layout

In many applications a large number of facilities are already preassigned, and only some new facilities are to be placed in such a way that the overall cost is minimized. In this case, the above algorithm can be applied with a few obvious modifications in the calculations. Since the preassigned objects and their locations will remain unchanged, these objects will always be in the set I and their locations will always be in the set J. In the search tree, if γ objects are already assigned, we start the search by assigning more objects, that is, the level of the tree starts at $\gamma + 1$. If the level of the tree ever becomes γ, then we stop.

3. SUBOPTIMAL AND OPTIMAL SOLUTIONS BY STEPPED FATHOMING

Due to the highly combinatorial nature of the problem, the task of finding an optimal solution and then verifying its optimality within a reasonable computational time is almost impossible in the case of large problems. Here we must resort to suboptimal solutions. The branch-and-bound procedure itself can be used to obtain qualified suboptimal solutions. In [1], Bazaraa and Elshafei proposed two stepped fathoming methods for obtaining controlled suboptimal solutions in the context of branch and bound. The application of these methods for the quadratic-assignment problem is discussed in this section.

Method 1

Recall that a partial solution is fathomed if the lower bound on all its completions is at least as big as $C^* > 0$, the best known objective value. Suppose that a partial solution is fathomed if $B \geq \alpha C^*$, where $\alpha \in (0,1]$. In this case, the partial solution is abandoned if there is no hope that it will lead to an objective which is better than αC^*. The purpose of this simple strategy is clear. We want to fathom the partial solution quickly even if it might lead to a slight improvement. Of course, as a new C^* is found, then we fathom whenever the bound is greater than or equal to α times the new C^*. The procedure continues until we cannot find a feasible solution with an objective less than αC^*. Thus, we have a feasible assignment with objective C^* coupled with the statement that the optimal objective is greater than or equal to αC^*.

Choice of α: Of course, if α is small, then fathoming will speed up considerably, resulting in a small computational effort. But on the other hand, the quality of the best feasible solution is not satisfactory. We recommend values of $\alpha \geq 0.9$, depending on the accuracy required.

Method 2

At each stage of the algorithm, we have an upper bound C^*. A lower bound on the overall problem L can be devised. Rather than fathoming on C^*, suppose we fathom on $K = \alpha C^* + (1 - \alpha)L$, where $\alpha \in (0,1]$. Since $L < C^*$, then $K \leq C^*$. Two cases are possible. In the first case, we will be able to find a complete solution with objective less than K. The objective value of this new solution becomes the new upper bound C^* and the process is repeated. In the second case, we will not be able to find such a solution. This automatically implies that there are no solutions with objective less than K and hence K itself is the new lower bound. The process is repeated. From this we keep narrowing the gap between the lower and upper bounds, either by lowering the upper bound when an improved feasible assignment is found or by raising the lower bound when no feasible solution with objective less than K is found. When the difference between the lower and upper bounds is smaller than a prescribed tolerance we stop.

Choice of α: Here α is any number in the interval $(0,1]$. Of course, if α is close to 1, then we are in effect fathoming on a number very close to C^*, and the search will not speed up considerably. On the other hand, if α is close to zero, then improvement is achieved only if we

obtain a feasible solution very close to the overall lower bound. In this case, fathoming will be fast, but it is likely not to obtain feasible solutions with an objective value that is less than K. If $\alpha = 0.5$ then the interval of uncertainty in which the optimal objective value lies will be halved at each stage.

Calculation of the Initial Overall Lower and Upper Bounds

Initial lower and upper bounds are needed to implement the above fathoming scheme. To calculate the lower bound, first calculate a lower bound b_{ij} on the cost of locating object i to location j, as discussed in Section 2. Then a linear-assignment problem is solved to find L. More precisely, let L be the optimal objective value of the following problem:

$$\text{Minimize} \sum_{i=1}^{m} \sum_{j=1}^{m} b_{ij} x_{ij}$$

$$\text{subject to} \sum_{j=1}^{m} x_{ij} = 1 \qquad \text{for } i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} x_{ij} = 1 \qquad \text{for } j = 1, \ldots, m,$$

$$x_{ij} \geq 0 \qquad \text{for } i, j = 1, \ldots, m.$$

We can obtain an upper bound C^* immediately by calculating the quadratic cost of the optimal assignment resulting from the above problem. Now Method 2 can be initiated.

Exact Solution by Stepped Fathoming

Either of the above two methods could be slightly modified to provide optimal solutions and still reduce the portion of the decision tree explicitly enumerated. Suppose that with any given α the search terminates with the conclusion that there exsits no feasible solution with a quadratic objective value less than K, where $K = \alpha C^*$ for the first method and $K = \alpha C^* + (1 - \alpha)L$ for the second method. The search can then be repeated from the complete stored solution whose objective value is C^* with a larger value of α.

Several increasing values of α, with the last value equal to one, could be used. Obviously, for $\alpha = 1$ we would fathom if the objective value is at least equal to C^*, and the method will produce an optimal solution. Even though portions of the decision tree may be repeated, the quick fathoming would usually result in a reduction of the overall computational effort. For further details, the reader is referred to [1].

4. COMPUTATIONAL EXPERIENCE

The experience gained with the solution procedure was in relation to the problems reported by Nugent, Vollman, and Ruml [13]. First, we will discuss some details pertinent to the purely computational aspects of the procedure.

Choice of α

If a small value of α is chosen, the fictitious upper bounds αC^* and $\alpha C^* + (1 - \alpha)L$ tend to be tighter, and hence the search becomes faster. However, we may increase the number of times we restart the search from the current best complete solution with a smaller value of α. On the other hand, if α is large then fathoming becomes weaker, and the search tends to be lengthier but to have fewer restarts. The tradeoff is only computational and is data dependent.

During the course of our study, we noticed that due to the features of the search procedure many successive good solutions are obtained very early in the search, and the optimum follows suit. As a result, we have adopted the strategy of choosing a small value of α at the beginning of the search and at a certain stage switching to $\alpha = 1$. We accomplish this by specifying an initial value of α and also specifying a difference between the actual upper and lower bounds at the achievement of which α is switched to 1. If the difference is appropriately chosen, we will get to the stage where the upper bound is tight enough to speed up the search, and also we will not have to restart the search once the tree is enumerated, as there is no interval of uncertainty in this case. We found that the choice of $\alpha = 0.7$ as a starting value was adequate for all the problems solved. The difference between the two bounds at which we switched to $\alpha = 1$ varied from one problem to another.

Solving the Linear Assignment Problem

As was mentioned in Section 2, the lower bound can be calculated by various methods. One procedure involves the use of a linear-assignment problem to obtain a tighter lower bound. There is no need, however, to solve a fresh assignment problem each time a lower bound is to be calculated. It is possible to take any previous solution and update it according to the new cost matrix.

Tables 1 and 2 summarize the experience with the following two codes:

QAP3: A code for an algorithm based on calculation of the lower bound $C_1 + C_2 + C_3$ by the matching of the ordered interaction and distance vectors as discussed in Section 2.

QAP7: A code for an algorithm based on calculation of the lower bound $C_1 + C_2 + C_3$ by the solution of a linear assignment problem. Here the cost matrix is first reduced so that it has a zero in every row and every column. If C^* is greater than or equal to the bound, we fathom. Otherwise, the complete linear-assignment problem is solved in the hope that the lower bound can be tightened.

Total Number of Nodes: the number of nodes, both intermediate and terminal, generated during the search.

Total Number of Moves: the number of forward and backward moves conducted during the search.

Number of times it was necessary to solve an assignment problem: whenever the lower bound calculated at any particular node by the reduction method was less than the current upper bound, it was necessary to solve a linear assignment problem to improve the value of this lower bound. Naturally, this strategy is applicable only to QAP7.

Fathoming Efficiency: the ratio between the number of times the search was not pursued, as a result of the lower-bound test, to the total number of times the lower-bound test was applied.

Comparison of QAP3 and QAP7: We recall that the only difference between QAP3 and QAP7 is the method of calculation of the lower bound as shown in Section 2. In Table 1, we notice that the solution times when QAP7 was used were always less than those obtained when QAP3 was used.

TABLE 1. *Summary of the Computational Experience with QAP3 and QAP7*

Problem Number	Size	Algorithm	Total Number of Nodes	Total Number of Moves	No. of Times the LB was Calculated	No. of Times it was Necessary to Solve an AP	Fathoming Efficiency %	Value of Best Solution Obtained	Solutions Time (seconds)*
4001	5×5	QAP3	15	38	43	–	46.12	25√	0.26
		QAP7	9	20	20	14	45.00	25√	0.15
4002	6×6	QAP3	27	99	129	–	54.26	43√	1.01
		QAP7	18	56	67	36	52.24	43√	0.63
4003	7×7	QAP3	55	177	235	–	51.06	74√	3.9
		QAP7	22	62	73	40	58.90	74√	2.7
4004	8×8	QAP3	231	739	1005	–	50.25	107√	15.3
		QAP7	52	179	235	141	52.77	107√	9.6
4005	12×12	QAP3	6877	29282†	–	–	–	289	221.8
		QAP7	5724	24496	37531	26368	50.02	289√	490.4
4006	15×15	QAP3	10071	40502‡	–	–	–	618	–

*On an IBM 370/165
†No restarting was allowed
‡The search was forced to termination
√Optimality verified

TABLE 2. *Number of Nodes Generated at the Tree Levels*

Problem Number	Algorithm	Number of Moves	Total Number of Nodes	Number of Nodes at Different Levels											
				1	2	3	4	5	6	7	8	9	10	11	12
4001	QAP3	38	15	0	4	5	3	3							
	QAP7	20	9	0	2	3	2	2							
4002	QAP3	99	27	0	6	13	4	2	2						
	QAP7	56	18	0	6	2	4	3	3						
4003	QAP3	177	55	0	7	21	21	3	1	2					
	QAP7	62	22	0	6	8	2	2	2	2					
4004	QAP3	739	231	0	8	40	109	58	10	3	3				
	QAP7	179	52	0	8	17	10	8	3	3	3				
4005	QAP3	29282*	6877	0	12	125	866	3367	1774	645	73	6	3	3	3
	QAP7	24496	5724	0	12	132	884	2539	1492	585	53	18	3	3	3

*Decision tree has not been exhausted.

Our observation about QAP3 is that it can face severe difficulties when the problem size increases. For example, problem 4005 was rerun with a starting upper bound equal to the true optimal objective value of $\alpha = 1$ in the hope that this would speed up the search. However, we had to terminate the problem after 50,400 moves, as we noticed that 11,539 nodes were generated but the number of nodes at various levels were:

0 4 41 358 2411 4101 3715 1000 184 8 3 3.

Thus a substantial part of the tree was still to be searched, and the estimated time for the completion of the search was about 15 minutes on the IBM 370/165. Also, the experience with 4006 was not more encouraging. Note that both QAP3 and QAP7 found the optimal solutions of problems 4001 through 4004 and verified optimality. QAP3 and QAP7 found the optimal solution of problem 4005, but only QAP7 verified optimality.

We might also add that the concept of stepped fathoming was essential in the procedure. QAP3 and QAP7 were not able to find optimal solutions to some of the reported problems when $\alpha = 1$ was used from the start of the search.

ACKNOWLEDGMENTS

The authors would like to thank Dr. L. F. McGinnis of the Georgia Institute of Technology for providing them with a code for the solution of the linear-assignment problem.

REFERENCES

[1] Bazaraa, M. S., and A. N. Elshafei, "On the Use of Fictitious Bounds in Tree Search Algorithms," Management Science *23*, 904-908 (1977).

[2] Breuer, M. A., "The Formulation of Some Allocation and Connection Problems as Integer Programs," Naval Research Logistics Quarterly *13*, 83-95 (1966).

[3] Dorris, A. L., "The Utility of Optimization Techniques in the Design of Man-Machine Systems," Masters Thesis, Georgia Institute of Technology, Atlanta, Georgia (1971).

[4] Elshafei, A. N., "Hospital Layout as a Quadratic Assignment Problem," Operational Research Quarterly *28*, 167-179 (1977).

[5] Gaschutz, G. K., and J. H. Ahrens, "Suboptimal Algorithm for the Quadratic Assignment Problem," Naval Research Logistics Quarterly *15*, 49-62 (1968).

[6] Gavett, J. W., and N. V. Plyter, "The Optimal Assignment of Facilities to Locations by Branch and Bound," Operations Research *14*, 210-232 (1966).

[7] Gilmore, P. C., "Optimal and Suboptimal Algorithms for the Quadratic Assignment Problem," SIAM Journal on Applied Mathematics *10*, 305-313 (1962).

[8] Graves, G. W., and A. B. Whinston, "An Algorithm for the Quadratic Assignment Problem," Management Science *17*, 453-471 (1970).

[9] Hopkins, L. D., "Land-Use Plan Design—Quadratic Assignment and Central Facility Models," Environment and Planning *9*, 625-642 (1977).

[10] Koopmans, T. C., and M. Beckman, "Assignment Problems and the Location of Economic Activities," Econometrica *25*, 53-76 (1957).

[11] Land, A. H., "A Problem of Assignment with Interrelated Costs," Operational Research Quarterly *14*, 185-198 (1963).

[12] Lawler, E. L., "The Quadratic Assignment Problem," Management Science *9*, 586-599 (1963).

[13] Nugent, C. E., T. E. Vollman, and J. Ruml, "An Experimental Comparison of Techniques for the Assignment of Facilities to Locations," Operations Research *16*, 150-173 (1968).

[14] Pierce, J. F., and W. B. Crowston, "Tree Search Algorithms for Quadratic Assignment Problems," Naval Research Logistics Quarterly *18*, 1-36 (1971).

[15] Steinberg, L., "The Backboard Wiring Problem: A Placement Algorithm," SIAM Journal on Applied Mathematics *3*, 37-50 (1961).

[16] Whitehead, B., and M. Z. Elders, "An Approach to the Optimum Layout of Single Story Buildings," Architect's Journal *139*, 1373-1380 (1964).

THE ENUMERATION OF ALL EFFICIENT SOLUTIONS FOR A LINEAR MULTIPLE-OBJECTIVE TRANSPORTATION PROBLEM

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ABSTRACT

An algorithm is presented by which the set of all efficient solutions for a linear multiple-objective transportation problem can be enumerated. First the algorithm determines an initial efficient basic solution. In a second step all efficient basic solutions are enumerated. Finally, the set of all efficient solutions is constructed as a union of a minimal number of convex sets of efficient solutions. The algorithm is illustrated by a numerical example.

1. INTRODUCTION

The classical transportation problem is a linear programming problem in which the constraints exhibit a particular type of mathematical structure. The usual scenario of the transportation-type constraints runs like this: At each of the m origins O_i there is a quantity a_i of a commodity at our disposal which we wish to ship to n destinations D_j to satisfy the demands b_j there. The symbol x_{ij} represents the unknown quantity shipped from O_i to D_j.

Instead of considering one scalar-valued objective function, this paper will focus on transportation problems with $k > 1$ linear objective functions to take care of those planning problems of economic origin which exhibit the mathematical structure of a transportation problem but are characterized by the existence of several objective functions. In most cases these functions are measured on different scales and have different units, and in general the decision maker is unable to combine these objective functions into one overall utility function. Let c_{ij}^l represent the proportional contribution to the value of the lth objective function of shipping one unit of the commodity from O_i to D_j. If the decision maker wants to minimize the k objective functions simultaneously, he will generally come to some point where a further reduction of the value of any objective function may only be obtained at the expense of increasing the value of at least one other objective function. In other words, at this point at least two of the k objective functions considered are in conflict.

Before going further, for convenience let us introduce the following notation. Let R denote the set of the real numbers, R_0 the set of the nonnegative real numbers, and R_+ the set of the positive real numbers. With regard to vector inequalities, the following convention will be applied: $\mathbf{a} \geqq \mathbf{b}$ if and only if $a_i \geqq b_i$ for all $i = 1, \ldots, n$; $\mathbf{a} \geqslant \mathbf{b}$ if and only if $a_i \geqq b_i$ for all $i = 1, \ldots, n$, and $a_i > b_i$ for at least one i; $\mathbf{a} > \mathbf{b}$ if and only if $a_i > b_i$ for all $i = 1, \ldots, n$. The transpose of a vector or a matrix will be denoted by an upper index (superscript) T.

The multiple-objective transportation problem is the problem of minimizing the k scalar-valued objective functions considered except for the conflicts among them. It may be stated as

$$\min z_1 = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^1 x_{ij},$$

$$\min z_2 = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^2 x_{ij},$$

$$\vdots$$

$$\min z_k = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^k x_{ij},$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \ldots, n,$$

$$x_{ij} \geq 0, \quad i = 1, \ldots, m; \; j = 1, \ldots, n,$$

or setting $M = \{1, \ldots, m\}$, $N = \{1, \ldots, n\}$, $J = \{(i, j) \mid i \in M, j \in N\}$, as the problem

$$\text{(1)} \qquad \text{"min"} \left\{ z = \sum_{(i,j) \in J} c_{ij} x_{ij} \;\middle|\; \begin{array}{ll} \sum_{j \in N} x_{ij} = a_i & \text{for all } i \in M, \\ \sum_{i \in M} x_{ij} = b_j & \text{for all } j \in N, \\ x_{ij} \geq 0 & \text{for all } (i,j) \in J, \end{array} \right\}$$

where $z \in R^k$ and $c_{ij} = c_{ij}^1, \ldots, c_{ij}^k)^T$. We shall assume throughout this paper $a_i > 0$, $b_j > 0$, and $\sum_{i \in M} a_i = \sum_{i \in N} b_j$. The set of all feasible solutions for (1) will be denoted by S. Let $x^0 = (x_{11}^0, x_{12}^0, \ldots, x_{mn}^0)$ be a feasible solution for (1). The solution x^0 is said to be an efficient or nondominated solution for (1) if and only if there is no other feasible solution x' for (1) such that $z' = \sum_{(i,j) \in J} c_{ij} x_{ij}' \leqq \sum_{(i,j) \in J} c_{ij} x_{ij}^0 = z^0$. In (1) the operator "min" indicates that all efficient solutions for (1) are to be determined. The set of all efficient solutions for (1) will be denoted by S^0. In general S^0 is not a convex set.

Thus, by solving the multiple-objective transportation problem (1) we end up with a subset of feasible solutions from among which we can be sure a most-preferred solution might lie, but between which, in the light of what is known about the decision maker's preference system at this point, no further discrimination is possible. Except for the case in which an efficient solution is minimal with respect to each scalar-valued objective function — which indicates that the objective functions are not in conflict — further information on the decision maker's preference system is necessary in order to select a most-preferred solution from the set of

efficient solutions. Though the algorithm we are going to present in this paper will be keyed to the determination of the set of all efficient solutions for the multiple-objective transportation problem, it is well suited to help the decision maker identify the relevant areas of efficient solutions and arrive at a final solution at which all other relevant efficient solutions have been considered and rejected.

In view of Theorem 1 in [3], a multiparametric programming routine could be applied to the multiple-objective transportation problem (1). A parametric-programming approach has been applied to bicriterion problems in [2] and [8]. As will be seen later, the proposed algorithm may be regarded as a dual method for parametric programming in that it identifies efficient solutions for (1) by an approach which is dual to that in multiparametric programming.

We shall distinguish three phases of the algorithm. In Phase I an initial efficient basic solution for the multiple-objective transportation problem (1) is found. Provided that the initial efficient basic solution is not unique, each of the other efficient basic solutions for (1) will be enumerated in Phase II. Finally, in Phase III the set of all efficient solutions for (1) is established as a union of a minimal number of convex sets of efficient solutions.

To solve the multiple-objective transportation problem, we shall utilize some duality results for multiple-objective linear programs and some fundamental results for the system of constraints in (1) which are the same as those need in ordinary transportation problems. Let $\mathbf{A}$ denote the matrix of coefficients of the linear constraints in (1), i.e., $\mathbf{A} = (a_{11}, \ldots, a_{mn})$, where $a_{ij} = e_i + e_{m+j}$ and $e_r \in R^{m+n}$ is a unit vector whose rth component is $+1$.

THEOREM 1: The matrix $\mathbf{A}$ is of rank $(m + n - 1)$.

THEOREM 2: The matrix $\mathbf{A}$ is totally unimodular, i.e., every square submatrix of $\mathbf{A}$ has its determinant equal to zero, $+1$, or -1.

Recall that a square matrix is said to be triangular if, after suitable rearrangement of rows and columns, all coefficients below the principal diagonal are zero.

THEOREM 3: Every basis of $\mathbf{A}$ is triangular.

A proof of the above theorems is found in [1] and [7].

A dual problem to (1) is the following (see [4] for details):

$$\text{(2)} \qquad \text{"max"} \left\{ g = \sum_{i \in M} u_i a_i + \sum_{j \in N} v_j b_j \;\middle|\; \sum_{(i,j) \in J} (u_i + v_j - c_{ij}) w_{ij} \geqslant 0 \right.$$
$$\left. \text{for no } w = (w_{11}, \ldots, w_{mn}) \in R_0^{n \cdot m} \right\},$$

where $u_i, v_j \in R^k$ are k-dimensional dual variables. The following duality result will be applied [4]:

THEOREM 4: Let $x^0 = (x_{11}^0, \ldots, x_{mn}^0)$ be a feasible solution for (1). The solution x^0 is an efficient solution for (1) if and only if there exists a feasible solution u_i^0 $(i = 1, \ldots, m)$, v_j^0 $(j = 1, \ldots, n)$ for the dual problem such that $\sum_{(i,j) \in J} c_{ij} x_{ij}^0 = \sum_{i \in M} u_i^0 a_i + \sum_{j \in N} v_j^0 b_j$. The solution u_i^0 $(i = 1, \ldots, m)$, v_j^0 $(j = 1, \ldots, n)$ is then itself an efficient solution for (2).

Since in (1) the system of constraints has the property that every basis is triangular, a feasible basic solution for (1) is easily obtained.

Let $x^0 = (x_{11}^0, \ldots, x_{mn}^0)$ be a feasible basic solution for (1) , and let J^0 denote the index set of the basic variables in x^0, i.e., $J^0 = \{(i, j) \in J \mid x_{ij}$ is a basic variable of the basic solution $x^0\}$. Now consider the system

$$c_{ij} = u_i + v_j \text{ for all } (i, j) \in J^0. \tag{3}$$

Since (3) is a system of $n + m - 1$ vector equations in $n + m$ vector-valued variables, we can assign an arbitrary value to one of the $m + n$ vector-valued simplex multipliers and then evaluate the remaining $n + m - 1$ multipliers thereby rendered unique. We shall assign a zero vector to u_1 and then evaluate the remaining $u_i (i = 2, \ldots, m)$ and $v_j (j = 1, \ldots, n)$. Let $u_i^0 (i = 1, \ldots, m)$, $v_j^0 (j = 1, \ldots, n)$ be the solution for the system (3). After multiplying the i th row equation in (1) by u_i^0 and the j th column equation by v_j^0 and summing up over i and j, respectively, we obtain

$$\sum_{i \in M} \sum_{j \in N} u_i^0 x_{ij} = \sum_{i \in M} u_i^0 a_i, \quad \sum_{j \in N} \sum_{i \in M} v_j^0 x_{ij} = \sum_{j \in N} v_j^0 b_j.$$

Let $z^0 = \sum_{(i,j) \in J} c_{ij} x_{ij}^0 = \sum_{i \in M} u_i^0 a_i + \sum_{j \in N} v_j^0 b_j$, $d_{ij}^0 = c_{ij} - u_i^0 - v_j^0$ for all $(i, j) \in J$.

The the vector-valued objective function can be written as

$$z = z^0 + \sum_{(i,j) \in J} d_{ij}^0 x_{ij}.$$

Recall that by (3) $d_{ij}^0 = 0$ if x_{ij} is a basic variable of x^0.

THEOREM 5: Let $x^0 = (x_{11}^0, \ldots, x_{mn}^0)^T$ be a feasible basic solution for (1). If the system

$$\sum_{(i,j) \in J} d_{ij}^0 w_{ij} \leqslant 0, \quad w_{ij} \in R_0 \text{ for } (i, j) \in J, \tag{4}$$

has no solution $w = (w_{11}, \ldots, w_{mn}$, then x^0 is an efficient solution for (1).

PROOF: Let the system (4) have no solution, which means that the $m + n$ vector-valued simplex multipliers u_i^0, v_j^0 are a feasible solution for the dual problem (2). Then there exists no $x \in S$ such that $\sum_{(i,j) \in J} d_{ij}^0 x_{ij} \leqslant 0$ holds. This implies, in connection with

$$\sum_{(i,j) \in J} d_{ij}^0 x_{ij} = \sum_{(i,j) \in J} c_{ij} x_{ij} - \sum_{i \in M} \sum_{j \in N} u_i^0 x_{ij} - \sum_{j \in N} \sum_{i \in M} v_j^0 x_{ij}$$

$$= \sum_{(i,j) \in J} c_{ij} x_{ij} - \sum_{(i,j) \in J} c_{ij} x_{ij}^0,$$

$$\sum_{(i,j) \in J} c_{ij} x_{ij} \leqslant \sum_{(i,j) \in J} c_{ij} x_{ij}^0 \text{ for no } x \in S \text{ and, hence, } x^0 \in S^0.$$

The converse of Theorem 5 will hold only for nondegenerate x^0.

THEOREM 6: Let $x^0 = (x_{11}^0, \ldots, x_{mn}^0)$ be an efficient nondegenerate basic solution for the multiple-objective transportation problem. Then the system (4) has no solution w.

PROOF: Since x^0 is nondegenerate, a unique basis and a unique set of reduced criterion vectors d_{ij}^0 $[(i,j) \in J]$ can be associated to x^0. Moreover, as x^0 is also efficient, according to Theorem 1 in [3] there exists a $\lambda^0 \in R_+^k$ such that $(\lambda^0)^T d_{ij}^0 \geqq 0$ holds for all $(i, j) \in J$. This implies, by Motzkin's theorem of the alternative ([6], p. 28), that the system (4) has no solution w.

An efficient basic solution for the multiple-objective transportation problem (1) is said to be dual feasible if and only if the system (4) has no solution w. Dual feasibility is a sufficient condition for the efficiency of a feasible basic solution for (1); it is, in addition to this, necessary for the efficiency of a nondegenerate feasible basic solution for (1). A degenerate feasible basic solution may be efficient without being dual feasible. However, Theorem 4 implies that from the set of all degenerate efficient basic solutions which have one efficient extreme point of S in common at least one degenerate efficient basic solution is dual feasible. Moreover, in view of Theorem 4, the determination of all dual feasible efficient solutions for (1) is adequate to construct the set S^0. For this reason, the proposed solution procedure will concentrate on determining all dual feasible efficient solutions for the multiple-objective transportation problem (1).

Recall that the matrix A of the multiple-objective transportation problem (1) is totally unimodular. If the coefficients a_i, for all $i \in M$, and b_j, for all $j \in N$, are integers, the values of the variables are integers in every feasible basic solution and, hence, in every efficient basic solution.

An alternate problem representation for the multiple-objective transportation problem (1) is the multiple-objective transportation tableau. The multiple-objective transportation tableau is, by definition, a rectangular tableau having m rows corresponding to the origins O_i and n columns corresponding to the destinations D_j. Each square (i, j) at the intersection of row i and column j contains quantities associated to the variable x_{ij}: the criterion vector c_{ij}, the value of the variable x_{ij}, and the reduced criterion vector d_{ij}. Moreover, the transportation tableau is bordered by a zero column containing the availabilities a_i, a zero row containing the demands b_j, a marginal column containing the vector-valued row multipliers, u_i, and a marginal row containing the vector-valued column multipliers, v_j.

A numerical example of a multiple-objective transportation problem with three objective functions (Figure 1) will be employed to illustrate the solution procedure. We shall apply a multiple-objective transportation tableau, to state the problem, and a starting feasible basic solution.

From this tableau we can observe that the present feasible basic solution x^0 is not dual feasible, because $d_{23}^0 < 0$.

2. PHASE I: DETERMINATION OF AN INITIAL EFFICIENT BASIC SOLUTION

In order to determine a first efficient solution for (1) we shall apply the following result:

THEOREM 7: At least one feasible basic solution for the multiple objective transportation problem (1) is an efficient solution.

PROOF: Recall that the set of feasible solutions for (1) is nonempty ([7], p. 226) and each feasible solution x for (1) can be expressed as a convex combination of the finitely many feasible basic solutions for (1) ([5], p. 145). Let $X = \{x^1, \ldots, x^r\}$ be the set of all feasible

	D_1 $b_1 = 60$		D_2 $b_2 = 80$		D_3 $b_3 = 160$		
$O_1 = a_1 = 100$	$c_{11} = \begin{pmatrix}3\\2\\7\end{pmatrix}$	$d''_{11} = \begin{pmatrix}9\\-1\\5\end{pmatrix}$	$c_{12} = \begin{pmatrix}1\\4\\7\end{pmatrix}$	$d''_{12} = \begin{pmatrix}9\\0\\-1\end{pmatrix}$	$c_{13} = \begin{pmatrix}-1\\3\\5\end{pmatrix}$		$u''_1 = \begin{pmatrix}0\\0\\0\end{pmatrix}$
					$x''_{13} = 100$		
$O_2 = a_2 = 125$	$c_{21} = \begin{pmatrix}4\\5\\1\end{pmatrix}$		$c_{22} = \begin{pmatrix}2\\6\\7\end{pmatrix}$		$c_{23} = \begin{pmatrix}5\\4\\1\end{pmatrix}$	$d''_{23} = \begin{pmatrix}-4\\-1\\-3\end{pmatrix}$	$u''_2 = \begin{pmatrix}10\\2\\-1\end{pmatrix}$
	$x^0_{21} = 45$		$x''_{22} = 80$				
$O_3 = a_3 = 75$	$c_{31} = \begin{pmatrix}-1\\3\\5\end{pmatrix}$		$c_{32} = \begin{pmatrix}6\\-1\\7\end{pmatrix}$	$d''_{32} = \begin{pmatrix}3\\-5\\-4\end{pmatrix}$	$c_{33} = \begin{pmatrix}4\\3\\8\end{pmatrix}$		$u''_3 = \begin{pmatrix}5\\0\\3\end{pmatrix}$
	$x''_{31} = 15$				$x''_{33} = 60$		
		$v''_1 = \begin{pmatrix}-6\\3\\2\end{pmatrix}$		$v''_2 = \begin{pmatrix}-8\\4\\8\end{pmatrix}$		$v''_3 = \begin{pmatrix}-1\\3\\5\end{pmatrix}$	$z'' = \begin{pmatrix}465\\1{,}230\\1{,}660\end{pmatrix}$

FIGURE 1. Multiple-objective transportation tableau 1

basic solutions for (1). Consider the set $Z = \{z^1, \ldots, z^r\}$, where $z^p = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x^p_{ij}$ for all $p = 1, \ldots, r$. Let z^s be a lexicographically minimal vector in Z, i.e., $z^s = \text{lexmin}\ \{z^1, \ldots, z^r\}$. Then x^s is an efficient solution for (1) since, for each $x^p \in X$, $z^p \leqq z^s$ does not hold and, because of the linearity of the vector-function z for each $x' \in S$, $z' = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x'_{ij} \leqq z^s$ does not hold either.

Applying the proof of Theorem 7, we can determine an initial efficient basic solution for the multiple-objective transportation problem (1) by solving the problem

$$\text{(5)} \qquad \text{lexmin} \left\{ z = \sum_{(i,j) \in J} c_{ij} x_{ij} \;\middle|\; \begin{array}{ll} \sum_{j=1}^{n} x_{ij} = a_i & \text{for all } i \in M, \\ \sum_{i=1}^{m} x_{ij} = b_j & \text{for all } j \in N, \\ x_{ij} \geq 0 & \text{for all } (i,j) \in J. \end{array} \right\}$$

Recall that a feasible solution for (1) which is a unique optimal solution with respect to the scalar-valued objective function z_1 is an optimal basic solution for (5) and, hence, an efficient basic solution for (1). However, if there is no unique optimal basic solution with respect to the scalar-valued objective function z_1, some of these optimal basic solutions may not be efficient solutions for (1). By the lexicographic-minimum problem (5), one feasible basic solution which is optimal with respect to z_1 is identified that is an efficient basic solution for (1).

Consider the transportation problem (5) and let x^0 be a feasible basic solution for (5) and u_i^0 $(i = 1, \ldots, m)$, $v_j^0 (j = 1, \ldots, n)$ be the associated simplex multipliers. If all relative criterion vectors d_{ij}^0 are lexicographically greater than or equal to the zero vector, i.e., $d_{ij}^0 = c_{ij} - u_i^0 - v_j^0 \gtrsim 0$ for all $(i, j) \in J$, then x^0 is an optimal solution for (5) and, hence, the initial efficient solution for the multiple-objective transportation problem (1). To show that the initial efficient solution for (1) thus determined is also dual feasible, let us assume, to the contrary, that the system (4) has a solution w^0. This implies that the zero vector is lexicographically greater than $\sum_{(i,j)\in J} d_{ij}^0 w_{ij}^0$ and, hence, also lexicographically greater than d_{ij}^0 for at least one pair $(i,j) \in J$, which is a contradiction of the optimality criterion applied above.

In Phase I of the multiple-objective transportation algorithm we have the following steps:

STEP 1: Determine an initial feasible basic solution in the multiple-objective transportation tableau.

STEP 2: Designate the set of pairs of indices (i,j) of the basic variables by I, and solve the system $u_1 = 0$, $u_i + v_j = c_{ij}$ for all $(i, j) \in I$.

STEP 3: Compute the relative criterion vector $d_{ij} = c_{ij} - u_i - v_j$ for all $(i,j) \in J \backslash I$.

STEP 4: Select $d_{st} = \text{lexmin}\{d_{ij} \mid (i, j) \in J \backslash I\}$.

STEP 5: If $d_{st} \gtrsim 0$, go to Step 7. Otherwise go to Step 6.

STEP 6: The variable x_{st} becomes a basic variable of the new feasible basic solution. Change the current solution to a new feasible basic solution and go to Step 2.

STEP 7: Designate the current feasible basic solution by x^1 and the corresponding index set I by J^1. The solution x^1 is an optimal solution for (5) and, hence, the initial efficient basic solution for (1). Store x^1, $z^1 = \sum_{(i,j)\in J} c_{ij} x_{ij}^1$, and J^1.

To illustrate Phase I of the multiple-objective transportation algorithm, we shall depart from the feasible basic solution of the multiple-objective transportation tableau 1. Since Steps 2 and 3 have been performed, we proceed with Step 4: $d_{23}^0 = \begin{bmatrix} -4 \\ -1 \\ -3 \end{bmatrix} = \text{lexmin}\{d_{ij}^0 \mid (i, j) \in J \backslash I\}$. As $d_{23}^0 \gtrsim 0$ does not hold, the current feasible basic solution does not satisfy the sufficient optimality criterion. The variable x_{23} becomes a basic variable of the new feasible basic solution, which is specified in multiple-object transportation tableau 2 (Figure 2).

Since $d_{21} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \text{lexmin}\{d_{ij} \mid (i, j) \in J \backslash I\}$ and $d_{21} \gtrsim 0$, the current feasible basic solution is optimal for (5) and is, hence, the initial efficient basic solution for (1). We shall designate this solution by x^1 and store x^1, z^1 and $J^1 = \{(1,3), (2,2), (2,3), (3,1), (3,3)\}$.

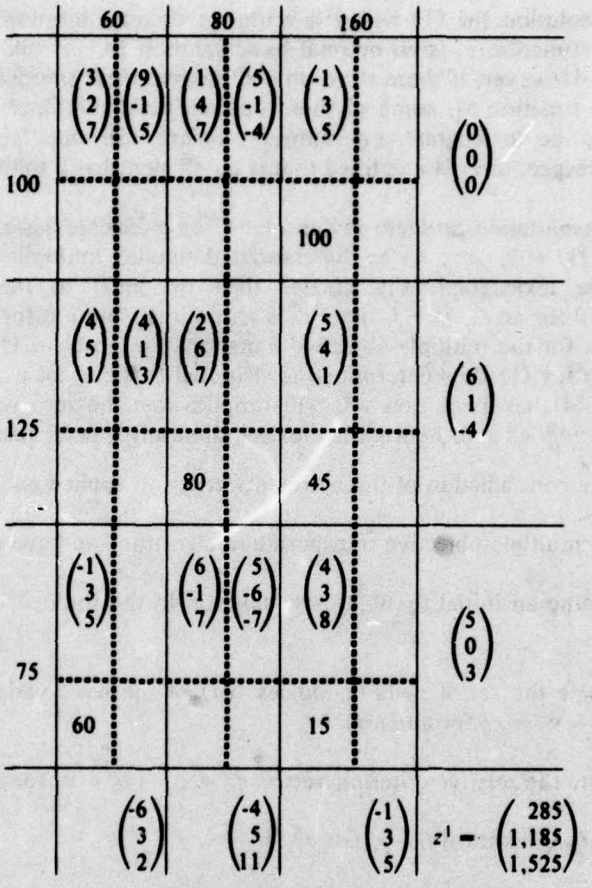

FIGURE 2. Multiple-objective transportation tableau 2

3. PHASE II: CONSTRUCTING THE SET OF ALL EFFICIENT BASIC SOLUTIONS

At the beginning of Phase II a dual feasible efficient basic solution x^σ for (1) and the associated multiple-objective transportation tableau are available. We shall now determine the remaining efficient basic solutions for (1) which are dual feasible. First let us introduce an adjacency relation between efficient basic solutions.

Let x' and x'' be efficient basic solutions for (1). Solutions x' and x'' are said to be adjacent if and only if (i) and (ii) hold:

(i) x' and x'' have $n+m-2$ basic variables in common;

(ii) each $x^0 = \alpha x' + (1-\alpha)x''$ $(0 \leq \alpha \leq 1)$ is an efficient solution for (1).

Let us consider an efficient basic solutions x^σ for the multiple-objective transportation problem (1) and the associated multiple-objective transportation tableau. In order to determine all dual feasible efficient basic solutions for (1) which are adjacent to x^σ we construct the set

$P^\sigma = \{(i,j) \in J \setminus J^\sigma \mid d_{ij} \ngeq 0\}$. Note that every feasible basic solution obtained by making x_{ij}, $(i,j) \notin P^\sigma$, a basic variable is not dual feasible and, if x_{ij} has been increased to a positive value, not efficient. Hence, if P^σ is empty, x^σ is the unique dual feasible efficient solution for (1). However, not every x_{ij} with $(i,j) \in P^\sigma$ is automatically an incoming variable of an adjacent efficient basic solution.

THEOREM 8: Let x^σ be a dual feasible efficient basic solution for (1) and Q^σ a nonempty subset of P^σ. If the linear program

$$(6) \qquad \max \left\{ 1^T y \;\middle|\; \sum_{(i,j) \in P^\sigma} d_{ij}^\sigma w_{ij} + Iy = 1, \quad y \in R_o^k, \; w_{ij} \geqq 0 \text{ for all } (i,j) \in P^\sigma \setminus Q^\sigma, \right\}$$

has an optimal solution, then each x_{ij}, $(i,j) \in Q^\sigma$, is an incoming variable of a dual feasible efficient basic solution for (1) which is adjacent to x^σ.

PROOF: Let (6) have an optimal solution $\hat{w}_{ij}$ $[(i,j) \in P^\sigma]$, $\hat{y}$. Then the linear program dual to (6),

$$(7) \qquad \min \left\{ 1^T \lambda \;\middle|\; \begin{array}{l} (d_{ij}^\sigma)^T \lambda \geqq 0 \text{ for all } (i,j) \in P^\sigma \setminus Q^\sigma, \\ (d_{ij}^\sigma)^T \lambda = 0 \text{ for all } (i,j) \in Q^\sigma, \\ \lambda \geqq 1, \end{array} \right\}$$

has an optimal solution $\hat{\lambda} \in R_+^k$. Not only x^σ, but also every feasible basic solution which is obtained from x^σ by changing x_{ij} $[(i,j) \in Q^\sigma]$ to a basic variable, is optimal for

$$(8) \qquad \min \left\{ \sum_{(i,j) \in J} \lambda^T c_{ij} x_{ij} \;\middle|\; \begin{array}{l} \sum_{j \in N} x_{ij} = a_i \text{ for all } i \in M, \\ \sum_{i \in M} x_{ij} = b_j \text{ for all } j \in N, \\ x_{ij} \geqq 0 \text{ for all } (i,j) \in J, \end{array} \right\}$$

with $\lambda = \hat{\lambda}$. Each of the feasible basic solutions thus enumerated is efficient, according to Theorem 1 in [3], and adjacent to x^σ. Applying Motzkin's theorem of the alternative (see, e.g., [6], p. 28) to the system $(d_{ij}^\sigma)^T \lambda \geqq 0$ for all $(i,j) \in P^\sigma$, we observe that each of the efficient basic solutions located is dual feasible. This completes the proof.

THEOREM 9: Let x^σ be a dual feasible efficient basic solution for (1) and x^ν be a dual feasible efficient basic solution for (1) which is adjacent to x^σ. Let $(s,t) \in P^\sigma$ denote the pair of indices of the incoming variable in order to proceed from x^σ to x^ν. Then there exists some $Q^\sigma \subseteq P^\sigma$ with $(s,t) \in Q^\sigma$ such that (6) has an optimal solution.

PROOF: The adjacency relation between x^σ and x^ν implies, in conjunction with Theorem 1 in [3], that for $Q^\sigma = \{(s,t)\}$ (7) has a feasible solution. As in (7) the objective function is bounded from below in the respective feasible set, (7) and hence also (6) have an optimal solution for $Q^\sigma = \{(s,t)\}$. This completes the proof.

Let H denote the index set of the dual feasible efficient basic solutions for (1). All $x^\sigma(\sigma \in H)$ and the existing adjacency relations can be represented by an undirected graph: Let $E = \{(x^\sigma | \sigma \in H\}$ and $L = \{(x^\sigma, x^\nu) \mid x^\sigma$ and x^ν are adjacent $(\sigma, \nu \in H)\}$. The undirected graph $G = (E,L)$ is said to be the solution graph associated with the multiple-objective transportation problem (1).

Theorems 8 and 9 establish the theoretical basis for identifying all x^ν $(\nu \in H)$ which are adjacent to any x^σ. Moreover, in connection with a fundamental property of the solution graph G, these two theorems also justify the proposed procedure to determine all $x^\sigma(\sigma \in H)$.

THEOREM 10: The solution graph G is finite and connected.

PROOF: Since the number of feasible basic solutions for (1) is finite, G is obviously finite. In order to see that G is connected, consider any pair (x^σ, x^ν) with $\sigma, \nu \in H$. According to Theorem 1 in [3] there exists a $\lambda^\sigma \in R_+^k$ $(\lambda^\nu \in R_+^k)$ such that x^σ (x^ν) is optimal for (8) with $\lambda = \lambda^\sigma$ $(\lambda = \lambda^\nu)$. By solving the one-parametric transportation problem

$$(9) \qquad \min \left\{ (\alpha\lambda^\sigma + (1-\alpha)\,\lambda^\nu)^T \sum_{(i,j)\in J} c_{ij}x_{ij} \;\middle|\; \begin{array}{l} \sum_{j\in N} x_{ij} = a_i \quad \text{for all } i \in M, \\ \sum_{i\in M} x_{ij} = b_j \quad \text{for all } j \in N, \\ x_{ij} \geq 0 \quad \text{for all } (i,j) \in J, \end{array} \right\}$$

for all α, $1 \geq \alpha \geq 0$, we obtain a sequence of efficient basic solutions $x^\sigma, x^{\sigma'}, \ldots, x^\nu$ where each two consecutive efficient basic solutions are adjacent. Hence each pair (x^σ, x^ν) of efficient basic solutions for (1) is linked by a chain in G. This completes the proof.

Consider an efficient basic solutions x^σ and let (6) have an optimal solution $\hat{w}_{ij}[(i,j) \in P^\sigma]$, $\hat{y}$ for $Q^\sigma = \{(s,t)\}$. If the pair of indices of each basic variable w_{ij} with $(i,j) \neq (s,t)$ of the given optimal solution for (6) is included in Q^σ, the initial optimal solution for (6) still remains optimal. In the course of the algorithm we shall enlarge the index set Q^σ in the proposed way, if possible, in order to reduce the number of linear programs (6) to be solved. Further investigations with respect to the index set Q^σ are necessary if we do not confine ourselves to the enumeration of all dual feasible efficient basic solutions for (1) but of the entire set S^0. We have to examine; whether or not further pairs of indices $(i,j) \in P^\sigma$ can be included in Q^σ such that the respective linear program (6) still has an optimal solution. Let $Q^\sigma \neq P^\sigma$ have less than $q = \min\{k, |P^\sigma|\}$ elements; then we successively examine whether or not by dropping the sign restriction of a nonbasic variable w_{ij} of the current optimal solution for (6) this variable can become a basic variable of an optimal solution for (6) in exchange for a basic variable y_l. If at the end of this procedure $Q^\sigma = P^\sigma$, Q^σ is obviously maximal. At the end of this procedure let $Q^\sigma \neq P^\sigma$. Then the pair of indices (i,j) of each nonbasic variable w_{ij} of the current optimal solution for (6) for which the reduced cost coefficient in the simplex tableau is zero has to be included in Q^σ in order to obtain a maximal index set Q^σ. Since the linear program (6) on which the described analysis is conducted has only k constraints, the computation in connection with the preceding analysis is easily carried out.

To ensure that for x^σ $(\sigma \in H)$ all maximal index sets Q^σ have been determined, further analysis is necessary. Let $T^\sigma = \{(s_1,t_1), \ldots, (s_g,t_g)\}$ denote the set of all pairs of indices which belong to at least one of the maximal sets Q^σ. All combinations of the g pairs of indices $(s_1,t_1), \ldots, (s_g,t_g)$ taken h pairs of indices at a time with $h = g, g-1, \ldots, 1$ can be represented in a directed graph $\Gamma(T^\sigma)$, in which the node which represents the combination

$< (s_1,t_1), \ldots, (s_g,t_g) >$ is the source and the $\binom{g}{1}$ nodes which represent all combinations of the g pairs of indices taken one pair of indices at a time are the sinks of $\Gamma(T^\sigma)$. Each node of $\Gamma(T^\sigma)$ represents a potential maximal set Q^σ. As one or more maximal sets Q^σ have already been determined, the graph $\Gamma(T^\sigma)$ has to be adjusted: For each of the already determined maximal index sets Q^σ the corresponding node, as well as all precessors and successors, will be deleted. If the adjusted graph $\Gamma(T^\sigma)$ has no node, all maximal index sets Q^σ have been identified. In order to identify further maximal index sets Q^σ or to make sure that no further maximal index set exists, we select a sink of the adjusted graph $\Gamma(T^\sigma)$ and solve the linear program (6) for the corresponding set Q^σ. If (6) has no optimal solution, the node which corresponds to Q^σ and all its precessors can be deleted. If, however, the program (6) has an optimal solution, a maximal set Q^σ will be determined and the corresponding node, all precessors, and all successors will be deleted.

We shall now utilize the above results and describe a systematic method to enumerate all dual feasible efficient basic solutions for (1) and, for each efficient basic solution x^σ, all maximal index sets Q^σ. Let σ be the index of that efficient basic solution currently under review; ν is the number of efficient basic solutions identified so far. The set $V^\sigma = \{\sigma+1, \ldots, \nu\}$ is the index set of the unexplored and $W^\sigma = \{1, \ldots, \sigma\}$ is the index set of the explored efficient basic solutions, including that one which is currently under review. Steps 9 to 23 of the algorithm apply in the main to the enumeration of all dual feasible efficient basic solutions.

STEP 8: Put $\sigma = 1$, $\nu = 1$, $V^0 = \{1\}$, and $W^0 = \emptyset$

STEP 9: Select x^σ and construct the associated multiple-objective transportation tableau. Put $V^\sigma = V^{\sigma-1} \backslash \{\sigma\}$, $W^\sigma = W^{\sigma-1} \cup \{\sigma\}$, and $T^\sigma = \emptyset$.

STEP 10: Construct the index set P^σ and put $P = P^\sigma$.

STEP 11: If $P = \emptyset$, go to Step 24. Otherwise go to Step 12.

STEP 12: Select some $(s,t) \in P$ and solve (6) for $Q^\sigma = \{(s,t)\}$.

STEP 13: If (6) has no optimal solution, put $P = P \backslash \{(s,t)\}$ and go to Step 11.

STEP 14: Include in Q^σ all pairs of indices $(i,j) \in P^\sigma$ such that Q^σ is a maximal set. Store Q^σ and put $T^\sigma = T^\sigma \cup Q^\sigma$.

STEP 15: Put $Q = P \cap Q^\sigma$ and $P = P \backslash Q$.

STEP 16: Select some $(s,t) \in Q$. Determine all $\bar{h}$ feasible basic solutions x^h, z^h, and the index sets J^h which can be constructed by making x_{st} a basic variable. Put $h = 0$.

STEP 17: Put $h = h + 1$

STEP 18: If there is any $\gamma \in V^\sigma \cup W^\sigma$, with $J^\gamma = J^h$, go to Step 19. Otherwise go to Step 22.

STEP 19: If $h = \bar{h}$ go to Step 20. Otherwise go to Step 17.

STEP 20: Put $Q = Q \backslash \{(s,t)\}$.

STEP 21: If $Q = \emptyset$ go to Step 11. Otherwise go to Step 16.

STEP 22: Put $\nu = \nu + 1$, $x^\nu = x^h$ and $V^\sigma = V^\sigma \cup \{\nu\}$. Store x^ν, J^ν and z^ν and go to Step 19.

STEP 23: If $\sigma = \nu$ print x^γ and z^γ for $\gamma = 1, \ldots, \nu$. Otherwise put $\sigma = \sigma + 1$ and go to Step 9.

The following steps apply to the construction of the graph $\Gamma(T^\sigma)$ and the process of identifying all further maximal sets Q^σ:

STEP 24: Construct the graph $\Gamma(T^\sigma)$ and adjust it according to the located maximal sets Q^σ.

STEP 25: If the adjusted graph $\Gamma(T^\sigma)$ has at least one node, go to Step 26. Otherwise go to Step 23.

STEP 26: Select a sink of $\Gamma(T^\sigma)$ and solve the linear program (6) for the corresponding set Q^σ.

STEP 27: If the linear program (6) has no optimal solution, adjust $\Gamma(T^\sigma)$ and go to Step 25. Otherwise go to Step 28.

STEP 28: Determine a maximal set Q^σ and adjust $\Gamma(T^\sigma)$. Store Q^σ and go to Step 25.

To illustrate the preceding steps of the multiple-objective transportation algorithm, we shall depart from the multiple-objective transportation tableau 2 with the initial efficient basic solution x^1. In Step 10 we construct $P^1 = \{(1,1), (1,2), (3,2)\}$ and put $P = P^1$. Since $P \neq \emptyset$, in Step 12 we select $(s,t) = (3,2)$ and solve the linear program (6), which becomes

$$\max \left\{ y_1 + y_2 + y_3 \;\middle|\; \begin{array}{rrrrrr} 9w_{11} & +5w_{12} & +5w_{32} + y_1 & & = 1 \\ -1w_{11} & -1w_{12} & -6w_{32} & +y_2 & = 1 \\ 5w_{11} & -4w_{12} & -7w_{32} & +y_3 & = 1 \\ & & w_{11}, w_{12}, y_1, y_2, y_3 & & \geq 0 \end{array} \right\}.$$

The optimal solution for this linear program is $\hat{w}_{11} = \hat{w}_{12} = \hat{y}_1 = 0$, $\hat{w}_{32} = 1/5$, $\hat{y}_2 = 11/5$, $\hat{y}_3 = 12/5$. If the pair of indices (1,1) or (1,2) is introduced into $Q^1 = \{(3,2)\}$, the linear program (6) has no optimal solution. Hence $Q^1 = \{(3,2)\}$ is a maximal index set, which will be stored. We obtain $T^1 = \{(3,2)\}$. In Step 15 we put $Q = \{(3,2)\}$ and $P = \{(1,1), (1,2)\}$. We pass to Step 16; x_{32} is the incoming variable. There is only one feasible basic solution to which x^1 can be revised: x_{32} is the new basic variable and x_{33} will become a nonbasic variable in the new efficient basic solution. Hence $\bar{h} = 1$. As this new efficient basic solution has not yet been identified, we continue with Step 22. We put $\nu = 2$, $V^1 = \{2\}$, and label the new efficient basic solution x^2. We store x^2, z^2, and $J^2 = \{(1,3), (2,2), (2,3), (3,1), (3,2)\}$. The values of x^2 and z^2 can be read from multiple-objective transportation tableau 3 (Figure 3).

Since $h = \bar{h} = 1$, and $Q = \emptyset$, but $P \neq \emptyset$, we continue with Step 12. For $Q^1 = \{(1,1)\}$ the linear program (6) has no optimal solution. Hence we put $P = \{(1,2)\}$. Also, for $Q^1 = \{(1,2)\}$ the linear program (6) has no optimal solution. As $P = \emptyset$, we proceed with Step 24 and construct the graph $\Gamma(T^1)$, which merely consists of the node $\{(3,2)\}$. $Q^1 = \{(3,2)\}$ has already been identified as a maximal index set; the corresponding node will be deleted. Since the adjusted graph $\Gamma(T^1)$ has no node we proceed with Step 23. As $P = \emptyset$ but $\sigma = 1 \neq \nu = 2$, we put $\sigma = 2$ and continue with Step 9.

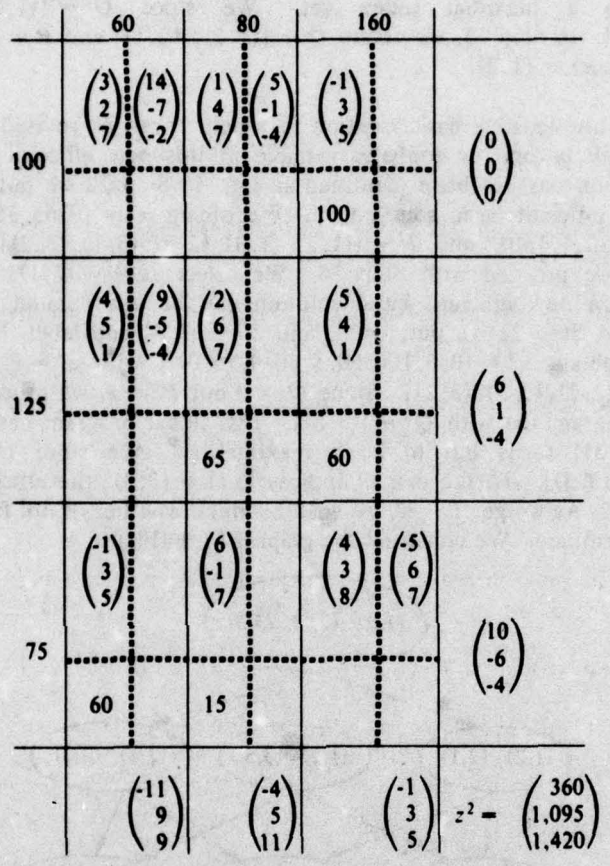

FIGURE 3. Multiple-objective transportation tableau 3

In Step 9 we put $V^2 = \emptyset$ and $W^2 = \{1,2\}$. From multiple-objective transportation tableau 3 we obtain $P^2 = \{(1,1), (1,2), (2,1), (3,3)\}$. We put $P = P^2$. In Step 12, we select $(s,t) = (1,1)$ and solve the linear program (6) for $Q^2 = \{(1,1)\}$. As this linear program has no optimal solution, we put $P = \{(1,2), (2,1), (3,3)\}$ and proceed with Step 12, since $P \neq \emptyset$. We select $(s,t) = (1,2)$ and solve the linear program (6) with $Q^2 = \{(1,2)\}$. The linear program (6) has an optimal solution which is specified by the following simplex tableau, with $w_{12} = w'_{12} - w''_{12}$ $(w'_{12}, w''_{12} \geq 0)$:

w_{11}	w_{12}	w''_{12}	w_{21}	w_{33}	y_1	y_2	y_3	
2.8	1	-1	1.8	-1	0.2			0.2
-4.2			-3.2	5	0.2	1		1.2
9.2			3.2	3	0.8		1	1.8
5				8				3

Obviously, $Q^2 = \{(1,2)\}$ is no maximal index set, since by dropping the sign restriction for w_{21}, the latter can be made a basic variable in exchange for y_2. The set $Q^2 = \{(1,2), (2,1)\}$

turns out to be a maximal index set. We store $Q^2 = \{(1,2), (2,1)\}$ and put $T^2 = \{(1,2),)(2,1)\}$. In Step 15, we obtain $Q = \{(1,2), (2,1)\}$ and $P = \{(3,3)\}$. We come to Step 16 and select $(s,t) = (1,2)$.

There is only one feasible basic solution to which x^2 can be revised; x_{12} is the new basic variable and x_{22} will become a nonbasic variable in this new efficient basic solution. This efficient basic solution has not been identified so far. In Step 22 we put $\nu = 3$, and $V^2 = \{3\}$ and label the new efficient basic solution x^3. We obtain $x^3 = (0,65,35,0,0,125,60,15,0)^T$, with $z^3 = (685,1,030,1,160)^T$ and $J^3 = \{(1,2),(1,3),(2,3),(3,1),(3,2)\}$. In step 20 we put $Q = \{(2,1)\}$, and we proceed with Step 16. We select $(s,t) = (2,1)$. With x_{21} being the incoming variable, a new efficient basic solution can be enumerated which has not been identified so far. In Step 22 we put $\nu = 4$, and $V^2 = \{3,4\}$ and label the new efficient basic solution x^4. We obtain $x^4 = (0,0,100,60,5,60,0,75,0)^T$, with $z^4 = (900, 795, 1,180)^T$ and $J^4 = \{(1,3),(2,1),(2,2),(2,3),(3,2)\}$. Since $Q = \emptyset$ but $P \neq \emptyset$, we proceed with Step 12 and solve the linear program (6) with $Q^2 = \{(3,3)\}$. This linear program has an optimal solution. The set $Q^2 = \{(3,3)\}$ turns out to be a maximal set. We store $Q^2 = \{(3,3)\}$ and put $T^2 = \{(1,2),(2,1),(3,3)\}$. To the pair of indices $(s,t) = (3,3)$, the efficient basic solution x^1 has to be associated. As we get $P = \emptyset$, we shall examine whether or not further maximal index sets Q^2 can be determined. We construct the graph $\Gamma(T^2)$ (Figure 4).

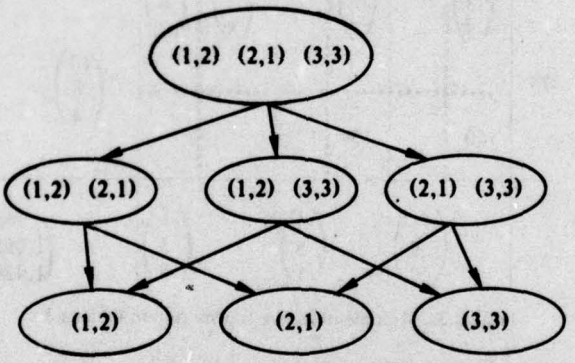

FIGURE 4. The graph $\Gamma(T^2)$

Since two maximal index sets Q^2 have been determined, $\Gamma(T^2)$ has to be adjusted accordingly. The adjusted graph $\Gamma(T^2)$ has no node; hence, all maximal index sets Q^2 have been determined. As $V^2 \neq \emptyset$ the algorithm of Phase II is continued with the exploration of x^3.

At the end of Phase II seven efficient basic solutions have been enumerated:

$x^1 = (0,0,100,0,80,45,60,0,15)^T$, $z^1 = (285, 1,185, 1,525)^T$, $J^1 = \{(1,3),(2,2),(2,3),(3,1),(3,3)\}$;
$x^2 = (0,0,100,0,65,60,60,15,0)^T$, $z^2 = (360, 1,095, 1,420)^T$, $J^2 = \{(1,3),(2,2),(2,3),(3,1),(3,2)\}$;
$x^3 = (0,65,35,0,0,125,60,15,0)^T$, $z^3 = (685, 1,030, 1,160)^T$, $J^3 = \{(1,2),(1,3),(2,3),(3,1),(3,2)\}$;
$x^4 = (0,0,100,60,5,60,0,75,0)^T$, $z^4 = (900, 795, 1,180)^T$, $J^4 = \{(1,3),(2,1),(2,2),(2,3),(3,2)\}$;
$x^5 = (0,5,95,60,0,65,0,75,0)^T$, $z^5 = (925, 790, 1,160)^T$, $J^5 = \{(1,2),(1,3),(2,1),(2,3),(3,2)\}$;
$x^6 = (60,0,40,0,5,120,0,75,0)^T$, $z^6 = (1,200, 675, 1,300)^T$, $J^6 = \{(1,1),(1,3),(2,2),(2,3),(3,2)\}$;
$x^7 = (60,5,35,0,0,125,0,75,0)^T$, $z^7 = (1,225, 670, 1,280)^T$, $J^7 = \{(1,1),(1,2),(1,3),(2,3),(3,2)\}$.

As shown by the solution graph G, the algorithm of Phase II may be regarded as enumerating all the nodes of a connected graph by examining all the neighbors of those nodes already enumerated. For the example considered the solution graph G may be depicted as shown in Figure 5.

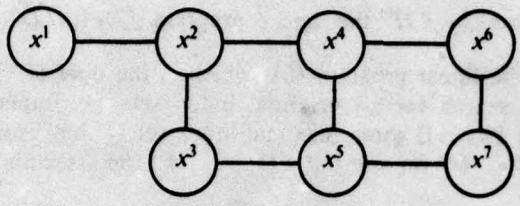

FIGURE 5. Solution graph G

THEOREM 11: Let x^γ be a dual feasible efficient basic solution for the multiple-objective transportation problem. Then x^γ has been determined at the conclusion of the described algorithm of Phase II.

PROOF: Let x^1 be the initial dual feasible efficient basic solution. If $J^1 \neq J^\gamma$, then by Theorem 10 there exists a sequence of adjacent $x^\sigma (\sigma \in H)$ linking x^1 with x^γ. The algorithm of Phase II uncovers each x^i of this sequence including x^γ.

4. PHASE III: ENUMERATION OF THE SET OF ALL EFFICIENT SOLUTIONS

In Phase II, for each efficient basic solution $x^\sigma (\sigma \in H)$ one or more maximal index sets Q^σ have been determined and stored. Let us form the index sets $R^\sigma = Q^\sigma \cup J^\sigma$ for each maximal index set Q^σ. If we consider the thus-determined index sets R^σ, we may discover the same index set several times and subsets of some index set R^σ as well. We shall form $\bar{\pi}$ index sets $U^\pi (\pi = 1, \ldots, \bar{\pi})$ from the sets R^σ with the following property: For each R^σ there exists some $\pi \in \{1, \ldots, \bar{\pi}\}$ such that $R^\sigma \subseteq U^\pi$, and for each U^π there exists at least one set R^σ such that $R^\sigma = U^\pi$. We further require $U^{\pi'} \subseteq U^{\pi''}$ for no $\pi' \neq \pi'' (\pi', \pi'' \in \{1, \ldots, \bar{\pi}\})$, and thus we obtain a minimal number $\bar{\pi}$ of sets U^π. Let $I^\pi = \{\sigma \in H \mid J^\sigma \subseteq U^\pi\}$. We are now in a position to assign to each set U^π a convex subset of the set of efficient solutions for the multiple-objective transportation problem (1),

$$S^\pi = \left\{ x \mid x = \sum_{\sigma \in I^\pi} \alpha_\sigma x^\sigma \left(\sum_{\sigma \in I^\pi} \alpha_\sigma = 1,\ \alpha_\sigma \geq 0 \text{ for all } \sigma \in I^\pi \right) \right\}.$$

THEOREM 12: S^π is a subset of S^0 for each $\pi = 1, \ldots, \bar{\pi}$.

PROOF: The construction of U^π ensures the existence of some $R^\sigma (\sigma \in H)$ such that $U^\pi = R^\sigma$. This implies that the linear program (6) with $Q^\sigma = R^\sigma \; J^\sigma$ has an optimal solution and also that the dual linear program (7) has an optimal solution λ^π. Each $x \in S^\pi$ is an optimal solution for (8) if $\lambda = \lambda^\pi$, and, according to Theorem 1 in [3], it is an efficient solution for the multiple-objective transportation problem (1). This completes the proof.

THEOREM 13: Let $x^0 = (x^0_{11}, \ldots, x^0_{mn})$ be an efficient solution for the multiple-objective transportation problem (1). Then there exists a $\pi \in \{1, \ldots, \bar{\pi}\}$ such that $x^0 \in S^\pi$.

PROOF: Let x^0 be an efficient solution for (1). According to Theorem 1 in [3], there exists a $\lambda^0 \in R^k_+$ such that x^0 is optimal for (8) if $\lambda = \lambda^0$. Let H^0 denote the index set of all dual feasible efficient basic solutions x^σ which are optimal for (8) when $\lambda = \lambda^0$. Then x^0 can be represented in the form

$$x^0 = \sum_{\sigma \in H^\sigma} \alpha_\sigma x^\sigma \left(\sum_{\sigma \in H^0} \alpha_\sigma = 1,\ \alpha_\sigma \geq 0 \text{ for all } \sigma \in H^0 \right).$$

Let $J^0 = \sum_{\sigma \in H^0} J^\sigma$. For some $\gamma \in H^0$ the linear program (7) for $Q^\gamma = Q^0 = J^0 \setminus J^\gamma$ has an optimal solution, so that the linear program (6), which is the dual of (7), also has an optimal solution. The systematic search for all maximal index sets Q^σ implies, in connection with Theorems 8 and 9, that in Phase II some maximal index set Q^γ has been constructed such that $Q^0 \subseteq Q^\gamma$. From $Q^\gamma \cup J^\gamma \subseteq U^\pi$ for some $\pi \in \{1, \ldots, \bar{\pi}\}$ the assertion follows.

The set of all efficient solutions for the multiple-objective transportation problem is then given by

$$S^0 = \bigcup_{\pi=1}^{\bar{\pi}} S^\pi.$$

Provided that all efficient basic solutions for (1) are nondegenerate $\bar{\pi}$ is the minimal number of convex sets of efficient solutions S^π. However, in the case of degeneracy it may happen that $S^{\pi'} \subseteq S^{\pi''}$ for some $\pi', \pi'' \in \{1, \ldots, \bar{\pi}\}$. In that case, the respective sets S^π are easily identified. We shall now present the iteration steps of Phase III.

STEP 29: Construct a minimal number $\bar{\pi}$ of index sets U^π from the complete list of index sets $R^\sigma = Q^\sigma \cup J^\sigma (\sigma \in H)$. Put $\pi = 0$.

STEP 30: Put $\pi = \pi + 1$.

STEP 31: Construct the index set I^π and the corresponding convex set of efficient solutions S^π.

STEP 32: If $\pi = \bar{\pi}$, go to Step 33. Otherwise go to Step 30.

STEP 33: Print the sets S^π for all $\pi = 1, \ldots, \bar{\pi}$.

To illustrate Phase III of the algorithm we shall now determine the sets S^π for our example. Recall that, for $\sigma = 1$, $\{(3, 2)\}$ is the only maximal index set Q^σ. We construct the index set $R^1 = \{(1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$. For $\sigma = 2$, two maximal index sets Q^σ have been determined: $\{(1, 2), (2, 1)\}$ and $\{(3, 3)\}$. We obtain two index sets R^2, of which the later is identical with R^1: $\{(1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2)\}$ and $\{(1, 3), (2, 2) (2,3), (3, 1), (3, 2), (3 ,3)\}$. From all sets R^σ which have been generated from the maximal index sets Q^σ $(\sigma = 1, \ldots, 7)$, $\bar{\pi} = 3$ index sets U^π can be constructed:

$U^1 = \{(1,3),(2,2),(2,3),(3,1),(3,2),(3,3)\}$;
$U^2 = \{(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2)\}$;
$U^3 = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,2)\}$.

The respective index sets I^π are: $I^1 = \{1,2\}$, $I^2 = \{2,3,4,5\}$, and $I^3 = \{4,5,6,7\}$. The set of all efficient solutions is given by

$$S^0 = \bigcup_{\pi=1}^{3} S^\pi,$$

where

$$S^1 = \left\{x \mid x = \sum_{\sigma=1}^{2} \alpha_\sigma x^\sigma \; (\alpha_1 + \alpha_2 = 1, \alpha_1, \alpha_2 \geq 0)\right\};$$

$$S^2 = \left\{x \mid x = \sum_{\sigma=2}^{5} \alpha_\sigma x^\sigma \left(\sum_{\sigma=2}^{5} \alpha_\sigma = 1; \alpha_\sigma \geq 0 \text{ for all } \sigma \in I^2 \right)\right\};$$

$$S^3 = \left\{x \mid x = \sum_{\sigma=4}^{7} \alpha_\sigma x^\sigma \left(\sum_{\sigma=4}^{7} \alpha_\sigma = 1, \; \alpha_\sigma \geq 0 \text{ for all } \sigma \in I^3 \right)\right\}.$$

REFERENCES

[1] DANTZIG, G.B., *Linear Programming and Extensions* (Princeton University Press, Princeton, New Jersey, 1963).

[2] GEOFFRION, A.M., "Solving Bicriterion Mathematical Programs," Operations Research **15**, 39-54 (1967).

[3] ISERMANN, H., "Proper Efficiency and the Linear Vector Maximum Problem," Operations Research **22**, 189-191 (1974).

[4] ISERMANN, H., "On Some Relations Between a Dual Pair of Multiple Objective Linear Programs," Zeitschrift fur Operations Research **22**, 33-41 (1978).

[5] LASDON, L. S., *Optimization Theory for Large Systems* (Macmillan, New York, 1970).

[6] MANGASARIAN, O.I., *Nonlinear Programming* (MacGraw-Hill, New York, 1969).

[7] SIMONNARD, M., *Linear Programming* (Prentice-Hall, Englewood Cliffs, N.J., 1966).

[8] SRINIVASAN, V., and G.L. THOMPSON, "Determining Cost vs. Time Pareto-Optimal Frontiers in Multi-Modal Transportation Problems," Transportation Science **11**, 1-19 (1977).

COORDINATED REPLENISHMENTS OF ITEMS UNDER TIME-VARYING DEMAND: DYNAMIC PROGRAMMING FORMULATION*

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ABSTRACT

We consider a group (or family) of items having deterministic, but time-varying, demand patterns. The group is defined by a setup-cost structure that makes coordination attractive (a major setup cost for each group replenishment regardless of how many of the items are involved). The problem is to determine the timing and sizes of the replenishments of all of the items so as to satisfy the demand out to a given horizon in a cost-minimizing fashion. A dynamic programming formulation is illustrated for the case of a two-item family. It is demonstrated that the dynamic programming approach is computationally reasonable, in an operational sense, only for small family sizes. For large families heuristic solution methods appear necessary.

1. INTRODUCTION

In this paper we consider a group (or family) of items for which the setup-cost structure is such that coordination of replenishments of the individual items is attractive. In addition, the individual-item demand patterns, although assumed deterministic, can vary with the time period. Contexts in which coordination is of interest include those cases in which a group of items share

(i) a common supplier;

(ii) a common mode of transportation (e.g., a freight car or an ocean-going container);

(iii) the same production facility.

Time-varying demand patterns permit us to treat a number of important situations including

(i) the fabrication of components and subassemblies in a multistage-production context — once the master schedules of finishing operations of the various end items are specified, the requirements of components and subassemblies through time are essentially deterministic and

*The research leading to this paper was supported by the National Research Council of Canada Grant No. A7417.
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time-varying (even for level end-item usage, the requirements for a particular component will vary with time because of a batching phenomenon, where the batches of different end items requiring the same component will likely be produced at different times);

(ii) production to a firm contract where the specified delivery quantities vary with time;

(iii) items with seasonal demand.

The single-item, time-varying problem has received considerable attention in the literature beginning with the fundamental dynamic programming treatment by Wagner and Whitin [23]. A number of authors (see, for example, Blackburn and Kunreuther [1], Crabill and Jaquette [3], Crowston and Wagner [4], Eppen et al. [8], Florian and Klein [9], Jagannathan and Rao [11], Kunreuther and Morton [12,13], Lambrecht [14], Love [15], Lundin and Morton [16], Newsom [17], Sawaki [18], Swoveland [21], Wagner [22], Zabel [24] and Zangwill [25]) have generalized the Wagner-Whitin model and associated results. In addition, Diegel [16] has treated the special case where the demand rate changes linearly with time. Several heuristic solutions have also been proposed for the basic Wagner-Whitin problem. Examples include De Matteis and Mendoza [5] and Silver and Meal [20].

For the case of *level* demand patterns and the coordinated setup-cost structure to be considered in the present paper, Goyal [10] has developed an iterative algorithm that finds the optimal solution within a particular class of policies. Brown [2] and Doll and Whybark [7] have also developed iterative solution procedures; however, Silver [19] has proposed a simpler, heuristic approach.

In Section 2, the physical process is described and the underlying assumptions of the model are laid out. Section 3 presents certain key properties of the optimal solution. For the case of just two items in the family, these properties are exploited to develop a dynamic programming formulation in Section 4. A numerical illustration is provided in Section 5. Section 6 is concerned with the general case of several items and the associated rapid growth in the number of states that have to be considered as a function of the number of items in the family and the number of time periods out to the chosen planning horizon. Finally, Section 7 provides some concluding remarks.

2. THE PHYSICAL PROCESS AND UNDERLYING ASSUMPTIONS

As discussed above, we consider a group of items where savings in replenishment costs are possible through the coordination of replenishments of two or more items. These items are faced with known, but likely time-varying, demand patterns out to some specified time horizon. The supply of each item is depleted according to its demand pattern. When the inventory of one or more items reaches a critically low level, a family replenishment, involving one or more items, is initiated. The problem is to select the timing and makeup of these replenishments in a cost-effective fashion.

The model that we shall develop is based on a number of assumptions, many of which could be relaxed (e.g., concave instead of linear carrying costs). We restrict attention to this set of assumptions because the problem was observed in this particular form within practical

contexts. In addition, although the dynamic programming approach could handle less-rigid assumptions, our ultimate goal is to develop simple, heuristic procedures, in which case the solution for the special situation considered (which is likely to be easier to achieve than the solution for the more general situation) would still be of substantial value to practitioners facing coordinated control under time-varying demand patterns.

The specific assumptions include

(1) We consider a family of n items where the family is defined according to the following setup-cost structure. There is a major setup cost A, in dollars, associated with any replenishment involving one or more of the items. There is a minor setup cost a_i, in dollars, incurred whenever item i is included in a replenishment of the family.

(2) The demand for item i is deterministic but varying with time. The symbol $D_{i,m}$ represents the demand for item i in period m ($i = 1, 2, \ldots, n$; $m = 1, 2, \ldots, N$, where N is the planning horizon). In the subsequent formulation, time will be measured in the conventional forward fashion.

(3) There are no quantity discounts and the unit-variable acquisition cost of any specific item is assumed constant, independent of the period of acquisition.

(4) The requirements for any period m must be on hand at or before the start of period m — this assumption was motivated by practical conditions in which the requirements are often input materials for further production; to provide short-range scheduling flexibility, all of the requirements for a period are needed at the start of the period.

(5) The initial inventory of each item is at the zero level, hence a group replenishment involving all items must take place at the start of period 1. (The extension to the more general case of nonzero initial inventories is straightforward).

(6) There is a carrying charge of h_i dollars/unit of item i carried from one period to the next (i.e., carrying charges are based on the period-ending inventories).

(7) We do not consider production constraints — this assumption is an obvious weakness in the production context. However, the problem is certainly of importance to distributors where production constraints are nonexistent.

As mentioned in the introduction, there has been considerable published material on single-item extensions of the basic Wagner-Whitin model. In addition, Crabill and Jaquette [3] treated a two-item problem having time-varying damands. However, their context did not include a coordinated setup-cost structure, but rather the added constraint that in any period at most one item can be replenished. Crabill and Jaquette, as well as the authors of all the other referenced material on the single-item problem, use less-restrictive assumptions (except for the key assumption of a coordinated setup-cost structure) than are utilized in the current paper.

3. PROPERTIES OF THE OPTIMAL SOLUTION

In this section we present four properties that the optimal policy must satisfy. Proofs of only the first and fourth are provided. The other two can be proved by methods identical to those already existing in the literature (see, e.g., Crabill and Jaquette [3]).

We first introduce some additional notation: Let $I_{i,m}$ be the inventory of item i at the *end* of period m *before* any delivery (production) at the start of period $m+1$, and let $Q_{i,m}$ be the amount of the replenishment of item i in (at the *start* of) period m.

PROPERTY 1: There exists an optimal solution (I^*,Q^*) such that

$$Q^*_{i,m} I^*_{i,m-1} = 0 \qquad \begin{matrix} \text{for } i = 1,2,\ldots,n, \\ m = 1,2,\ldots,N. \end{matrix}$$

In words, we never replenish an item until its inventory drops to zero. This is a property of the basic Wagner-Whitin model. If the starting inventory of an item is nonzero, then this property need not hold for *only* the first replenishment.

PROOF: Consider any solution (Q,I) such that $Q_{i,m}\, I_{i,m-1} > 0$ for at least one i,m pair.

Let $k = \min\,(m \mid Q_{i,m}\, I_{i,m} > 0)$.

In case there is more than one i with $Q_{i,k}\, I_{i,k-1} > 0$, let

$$j = \min\,(i \mid Q_{i,k}\, I_{i,k-1} > 0)$$

and t be the time of the previous replenishment of product j.

Consider an alternative policy A obtained from (Q,I), as follows: The inventory of (Q,I) at the end of period $k-1$ is subtracted from the replenishment quantity in period t and added to the replenishment quantity at the start of period k. Mathematically, we have

$$\begin{aligned}
Q^A_{j,t} &= Q_{j,t} - I_{j,k-1}, \\
Q^A_{j,k} &= Q_{j,k} + I_{j,k-1}, \\
I^A_{j,m} &= I_{j,m} - I_{j,k-1}, \qquad m = t,t+1,\ldots,k-1, \\
Q^A_{i,m} &= Q_{i,m}, \qquad \text{all other } i,m, \\
I^A_{i,m} &= I_{i,m}, \qquad \text{all other } i,m.
\end{aligned}$$

From assumptions (3) and (6) it follows that

$$\text{Costs } (Q,I) - \text{Costs } (Q^A,I^A) = I_{j,k-1}(k-t)h_j > 0,$$

i.e., policy A improves on the performance of the initial policy. Furthermore, policy A has

$$Q^A_{i,m}\, I^A_{i,m-1} = 0 \quad \text{for } m \leqslant k,\ i \leqslant j.$$

This reasoning is repeated, if necessary, for $m = k$, $i > j$, and then, if necessary, for $m > k$, all i. Thus we eventually end with a solution satisfying Property 1 that costs less than policy (Q,I).

PROPERTY 2: If we restrict ourselves to policies satisfying Property 1, the order quantity placed for item i at the start of period m must be one of

$$D_{i,m},\ D_{i,m} + D_{i,m+1},\ \ldots,\ \sum_{r=m}^{N} D_{i,r}.$$

PROPERTY 3: Again, restriction to policies satisfying Property 1 (or 2) implies that $I_{i,m-1}$ must be one of 0, $D_{i,m}$, $D_{i,m} + D_{i,m+1}$, ..., $\sum_{r=m}^{N} D_{i,r}$.

PROPERTY 4 (Planning Horizon Concept): Consider $t < m$. If we have

$$(m-t)h_i D_{i,m} > A + a_i, \tag{1}$$

then the requirements $D_{i,m}$ (and those for any later period for item i) would not be included in an order placed at the start of period t (namely $Q_{i,t}$).

PROOF: $(m-t)h_i D_{i,m}$ represents the carrying costs associated with the requirements $D_{i,m}$ if they are included in a replenishment at the start of period t. If inequality (1) holds, this implies that it costs less to make a special group replenishment, involving* only item i, at the start of period m. Thus, any solution in which inequality (1) was violated could be improved upon by removing $D_{i,m}$ from the replenishment at the start of period t and instead placing it in a replenishment at the start of period m.

4. DYNAMIC PROGRAMMING FORMULATION FOR THE CASE OF TWO ITEMS†

Let (j_1, j_2) represent the inventory state (before a replenishment) at the start of a particular period where j_1 is the number of periods of supply of item 1, and j_2 is the number of periods of supply of item 2. Then by Properties 1 and 3 we know that we can restrict the states from which we place an order to those included in three sets:

(i) (j_1 = positive integer, $j_2 = 0$);

(ii) ($j_1 = 0$, j_2 = positive integer);

(iii) ($j_1 = 0$, $j_2 = 0$).

Now let $f_n(j,0)$ be the *minimum* cost in periods $n, n+1, \ldots, N$ if we are at the start of period n (prior to an order) with $j > 0$ periods of supply left for item 1 and 0 periods left for item 2, $f_n(0,j)$ be the same but with items 1 and 2 interchanged, and $f_n(0,0)$ be the minimum cost in periods $n, n+1, \ldots, N$ if we are at the start of period n and both items must be ordered. When item 1 is ordered we shall let the decision variable be T_1, the replenishment quantity, expressed as a number of periods of supply (we know from Property 2 that $T_1 = 1, 2, 3, \ldots$). T_2 will be the similar quantity for item 2.

We shall illustrate the development of the recursion relationship for the case of a state $(j,0)$ with $j > 0$. The state $(j,0)$ with $j > 0$ at the start of period n implies that only item 2 has to be replenished at the start of that period. Both the state variable j and the decision variable T_2 can take on any of the values $1, 2, \ldots, N-n+1$. (Of course, for a specific numerical example use of Property 4 could reduce the range of one or both of the variables.) The *next* ordering state (after the replenishment at the start of period n) depends upon the relative values of j and T_2. There are two possibilities:

*There may possibly be a tighter horizon property than Property 4, in that a replenishment of item i at the start of period m need not incur the full major cost A (the requirements of another item may dictate a group replenishment at that time).

†A numerical illustration of the solution procedure will be shown for a six-period problem in Section 5.

CASE 1: if $T_2 \leqslant j$, item 2 again runs out before* item 1 and the next ordering state is $(j-T_2, 0)$ at the start of period $n+T_2$.

CASE 2: if $T_2 \geqslant j$, item 1 runs out before* item 2 and the next ordering state is $(0, T_2-j)$ at the start of period $n+j$. Thus, we have,† for $n = 2, 3, \ldots, N-1$, $j = 1, 2, \ldots, N-n+1$,

$$f_n(j,0) = A + a_2 + \min_{T_2 = 1,2,\ldots,N-n+1} B(j,T_2), \tag{2}$$

where $A + a_2$ represents the setup costs of a replenishment involving only item 2, and $B(j,T_2)$ represents the carrying cost from the start of period n to the time of the next replenishment plus the functional value at that future time.

We have, for $T_2 \leqslant j$,

$$B(j,T_2) = h_1 \sum_{i=1}^{T_2} (i-1) D_{1,n+i-1} + h_1 T_2 \sum_{i=T_2+1}^{j} D_{1,n+i-1} + h_2 \sum_{i=1}^{T_2} (i-1) D_{2,n+i-1} + f_{n+T_2}(j-T_2, 0) \tag{3}$$

and, for $T_2 \geqslant j$,

$$B(j,T_2) = h_1 \sum_{i=1}^{j} (i-1) D_{1,n+i-1} + h_2 \sum_{i=1}^{j} (i-1) D_{2,n+i-1} + h_2 j \sum_{i=j+1}^{T_2} D_{2,n+i-1} + f_{n+j}(0, T_2 - j). \tag{4}$$

In expression (3) the first summation term expresses the carrying costs (from the start of period n) of the requirements of item 1 for periods $n+1$ to $n+T_2-1$ inclusive, the second summation term‡ represents the cost of carrying the requirements of periods $n+T_2$, $n+T_2+1, \ldots, n+j$ for item 1 through the T_2 periods until the time of the next replenishment (at the start of period $n+T_2$), and the third summation term gives the carrying costs (from the start of period n) for the requirements of item 2 for periods $n+1$ to $n+T_2-1$, inclusive. Of course, the last term represents the functional value of being in the next ordering state, $(j-T_2, 0)$, at the time of the next order, the start of period $n+T_2$. The terms in expression (4) can be similarly interpreted when one realizes (for $T_2 \geqslant j$) that item 1 will now run out at the start of period $n+j$ when there will still be T_2-j periods of supply of item 2 remaining. Recursion relationships, closely paralleling those of equations (2), (3), and (4), can be developed for the other two types of ordering states.

Finally, the boundary conditions are

$$f_N(0,0) = A + a_1 + a_2, \tag{5}$$

$$f_N(1,0) = A + a_2, \tag{6}$$

and

$$f_N(0,1) = A + a_1. \tag{7}$$

*If $T_2 = j$, replace "again runs out before" or "runs out before" by "runs out at the same time as".

†For $n=1$ the only state that need be considered is the given initial state, namely (0,0). In addition, a ste of boundary conditions will be specified for the last period ($n=N$).

‡The summation is defined to be zero when $T_2=j$.

5. NUMERICAL ILLUSTRATION WITH TWO ITEMS (n=2)

We consider a six-period horizon (N=6) with the two demand patterns as shown in Table 1.

TABLE 1

Period, m	$D_{1,m}$	$D_{2,m}$
1	10	10
2	70	10
3	80	20
4	5	40
5	5	10
6	70	10

Other relevant characteristics are A = \$40, a_1 = \$10, a_2 = \$20; h_1 = \$3/unit/period; and h_2 = \$1/unit/period.

The various possible states (taking advantage of Property 4), their associated functional values, and the optimal values of the decision variables are shown in Table 2. From the table we trace back the solution as follows: The initial state is (0,0) at n=1. From the last line of the table we see that the initial T's are T_1=1 and T_2=2. This leads to the next ordering state being (0,1) at n=2. Now, the table gives T_1=1. Hence, the next ordering state is (0,0) at n=3. The appropriate row of the table indicates that T_1=1, T_2=2. Continuing in this fashion we develop the entire solution shown in Table 3.

6. THE GENERAL CASE OF n ITEMS

We saw in the previous section that for the case of two items we had to consider the maximum number of states (ie., ignoring any potential benefits of Property 4) shown in Table 4. Thus, the total number of states is given by

$$\sum_{k=1}^{N-1} (2k+1) + 1,$$

which simplifies to N^2.

Now, when we go to $n > 1$ items, conceptually the approach does not change. However, the number of possible states proliferates. When there are k periods remaining, the possible ordering states include: only 1 item is ordered (there are n choices of the item and each of the other items can be in any one of k possible states), 2 items are ordered (there are* n_{C_2} choices of the 2 items and each of the remaining items can be in any one of k possible states), and others.

If we let $F(n,N)$ represent the total number of states (ignoring the effects of Property 4), then a continuation of the above argument yields

*$n_{C_2} = \frac{n!}{2!(n-2)!}$

$$F(n,N) = 1\# + \sum_{j=1}^{N-1} (n\,j^{n-1} + {}^{n}C_2\,j^{n-2} + \ldots + n\,j + 1)$$

$$= 1 + \sum_{j=1}^{N-1} [({}^{n}C_0 j^n + {}^{n}C_1 j^{n-1} + {}^{n}C_2 j^{n-2} + \ldots + {}^{n}C_{n-1} j^1 + {}^{n}C_n j^0) - {}^{n}C_0 j^n] \quad \begin{pmatrix}\text{adding and}\\ \text{subtracting } {}^{n}C_0 j^n\end{pmatrix}$$

$$= 1 + \sum_{j=1}^{N-1} [(1+j)^n - j^n]$$

$$= 1 + [2^n - 1^n] + [3^n - 2^n] + \ldots + [N^n - (N-1)^n]$$

$$= N^n.$$

Thus, dynamic programming is only feasible for fairly small values of n and N. Perhaps the equivalent formulation as a shortest-route problem might extend the range of feasible computation somewhat in that, as discussed by Crabill and Jaquette [3], a properly encoded shortest-route formulation is computationally more efficient than dynamic programming.

TABLE 2. *Details of the Solution of the Numerical Example*

n	State for Item 1, j_1	State for Item 2, j_2	Best T Values for Given State and Stage		Functional Value
			$T^*_1(0, j_2)$	$T^*_2(0, j_2)$	$f_n(0, j_2)$
6	1	0	–	1	60
	0	1	1	–	50
	0	0	1	1	70
5	1	0	–	2	120
	0	2	1	–	110
	0	1	1	–	120
	0	0	1	2	130
4	2	0	–	1	155
	1	0	–	1,2,3	190
	0	3	2	–	145
	0	2	2	–	145
	0	1	1	–	180
	0	0	2	2,3	165
3	3	0	–	3	235
	2	0	–	2	245
	1	0	–	1	225
	0	4	3	–	235
	0	3	3	–	225
	0	2	2	–	235
	0	1	1	–	215
	0	0	1	1	235
2	1	0	–	1,2	295
	0	2	1	–	285
	0	1	1	–	285
	0	0	1	1,2	305
1	0	0	1	2	365

In the first period there is only a single state that need be considered because of the prescribed initial conditions.

TABLE 3. *Summary of the Solution to the Numerical Example* ($A = \$40$, $a_1 = \$10$, $a_2 = \$20$, $h_1 = \$3$, $h_2 = \$1$)

Period m	Item 1				Item 2				Setup Cost
	$T_{1,m}$	$Q_{1,m}$	$D_{1,m}$	$I_{1,m}$	$T_{2,m}$	$Q_{2,m}$	$D_{2,m}$	$I_{2,m}$	
1	1	10	10	0	2	20	10	10	\$70
2	1	70	70	0	—	—	10	0	50
3	1	80	80	0	1	20	20	0	70
4	2	10	5	5	3	60*	40	20	70
5	—	—	5	0	—	—	10	10	—
6	1	70	70	0	—	—*	10	0	50
Total				5				40	\$310

Total costs = 310 + 3(5) + 1(40)

= \$365.

*There is an alternative optimal solution with $Q_{2,4} = 50$ instead of 60 and $Q_{2,6} = 10$ instead of 0.

TABLE 4

Number of periods Remaining, k	Types of States	Total Number of States
1	(1,0),(0,1)(0,0)	3
2	(2,0),(1,0),(0,2),(0,1),(0,0)	5
.	.	.
.	.	.
.	.	.
$k < N$	$(j > 0,0),(0,j > 0),(0,0)$	$2k + 1$
N	initial state (0,0)	1

7. CONCLUSIONS

In this paper we have addressed a difficult inventory-control problem, namely that of a group of items having time-varying demand patterns and being linked by a coordinated setup-cost structure. As discussed in the introduction, there are a number of important practical contexts that can be represented in such a problem format. A dynamic programming solution method, suitable for small groups of items, has been developed and illustrated. This solution approach is thus directly applicable to small-scale versions of the aforementioned realistic problems. The demonstrated proliferation of the state space for larger groups of items indicates that heuristic procedures are needed (a topic of ongoing research). Nevertheless, for the testing of such heuristics, it may be reasonable to obtain optimal solutions by dynamic programming for a few test examples with somewhat larger values of n.

Important extensions of the basic model would include the introduction of capacity restrictions and/or quantity-discount possibilities. Either of these extensions dramatically complicates the class of solutions that must be considered because Property 1 need no longer be satisfied by the optimal solution.

ACKNOWLEDGMENT

The author would like to acknowledge several suggestions made by an anonymous referee that have made the presentation more lucid.

REFERENCES

[1] Blackburn, J., and H. Kunreuther, "Planning and Forecast Horizons for the Dynamic Lot Size Model with Backlogging," Management Science *21*, 251-255 (1974).

[2] Brown, R.G., *Decision Rules for Inventory Management* (Holt, Rinehart and Winston, New York, 1967), Chap. 5.

[3] Crabill, T., and D. Jaquette, "A Two Product Dynamic Economic Lot Size Production Model with Either-Or-Production Constratints," Naval Research Logistics Quarterly *21*, 505-513 (1974).

[4] Crowston, W.B., and M.B. Wagner, "Dynamic Lot-Size Models for Multi-Stage Assembly Systems", Management Science *20*, 14-21 (1973).

[5] De Matteis, J.J., and A.G. Mendoza, "An Economic Lot-Sizing Technique," IBM Systems Journal *7*, 30-46 (1968).

[6] Diegel, A., "A Linear Approach to the Dynamic Inventory Problem," Management Science *12*, 530-540 (1966).

[7] Doll, C.L., and D.C. Whybark, "An Iterative Procedure for the Single-Machine, Multi-Product, Lot Scheduling Problem," Management Science *20*, 50-55 (1973).

[8] Eppen, G.D., F.J. Gould, and B.P. Pashigian, "Extensions of the Planning Horizon Theorem in the Dynamic Lot Size Model," Management Science *15*, 268-277 (1969).

[9] Florian, M., and M. Klein, "Deterministic Production Planning with Concave Costs and Capacity Constraints," Management Science *18*, 12-20 (1971).

[10] Goyal, S.K., "Determination of Optimum Packaging Frequency of Items Jointly Replenished," Management Science *21*, 436-443 (1974).

[11] Jagannathan, R., and R. Rao, "A Class of Deterministic Production Planning Models," Management Science *19*, 1295-1300 (1973).

[12] Kunreuther, H., and T. Morton, "Planning Horizons for Production Smoothing with Deterministic Demands: I," Management Science *20*, 110-125 (1973).

[13] Kunreuther, H., and T. Morton, "Planning Horizons for Production Smoothing with Deterministic Demands: II," Management Science *20*, 1037-1046 (1974).

[14] Lambrecht, M., "Capacity Constrained Multi-Facility Dynamic Lot-Size Problem", Nummer 19, Faculteit der Economische en Toegepaste Economische Wetenschappen, Katholieke Universiteit Leuven, Leuven, Belgium (1976).

[15] Love, S.F., "Bounded Production and Inventory Models with Piecewise Concave Costs," Management Science *20*, 313-318 (1973).

[16] Lundin, R., and T. Morton, "Planning Horizons for the Dynamic Lot Size Model: Zabel vs Protective Procedures and Computational Results," Operations Research *23*, 711-734 (1975).

[17] Newsom, P., "Multi-Item Lot-Size Scheduling by Heuristic, Part 1: With Fixed Resources," Management Science *21*, 1186-1193 (1975).

[18] Sawaki, K., "Some Extensions of Dynamic Economic Lot-Size Model with Backlogging," Journal of the Operations Research Society of Japan *14*, 1-18 (1971).

[19] Silver, E.A., "A Simple Method of Determining Order Quantities in Joint Replenishments under Deterministic Demand," Management Science *22*, 1351-1361 (1976).

[20] Silver, E.A., and H.C. Meal, "A Heuristic for Selecting Lot Size Requirements for the Case of a Deterministic Time-Varying Demand Rate and Discrete Opportunities for Replenishment," Production and Inventory Management *14*, 64-74 (1973).

[21] Swoveland, C., "A Deterministic Multi-Period Production Planning Model with Piecewise Concave Production and Holding-Backorder Costs," Management Science *21*, 1007-1013 (1975).

[22] Wagner, H., "A Postscript to 'Dynamic Problems in the Theory of the Firm'," Naval Research Logistics Quarterly *7*, 7-12 (1960).

[23] Wagner, H., and T.M. Whitin, "Dynamic Version of the Economic Lot Size Model," Management Science *5*, 89-96 (1958).

[24] Zabel, E., "Some Generalizations of an Inventory Planning Horizon Theorem," Management Science *10*, 465-471 (1964).

[25] Zangwill, W.I., "A Backlogging Model and a Multi-Echelon Model of a Dynamic Economic Lot Size Production System — A Network Approach," Management Science *15*, 506-527 (1969).

A NOTE ON DUALITY IN HOMOGENEOUS FRACTIONAL PROGRAMMING

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ABSTRACT

For a linear fractional programming problem, Sharma and Swarup have constructed a dual problem, also a linear fractional program, in which the objective functions of both primal and dual problems are the same. Craven and Mond have extended this result to a nonlinear fractional programming problem with linear constraints, and a dual problem for which the objective function is the same as that of the primal. This theorem is now further extended from linear to differentiable convex constraints.

Duality for fractional and nonlinear programming has been treated extensively in the recent literature — see Schaible [4], where many references are given. Sharma and Swarup [5] gave a formulation of linear fractional programming where the dual problem is a fractional program with the same objective function as the primal. Craven and Mond [1] extended this to a nonlinear fractional program with linear constraints. The result is now extended to differentiable convex constraints.

Consider the two fractional programming problems:

$$\text{(P):} \qquad \underset{x}{\text{Maximize}} \quad \{f(x)/g(x) : h(x) \leqslant 0,\ x \in X_0\};$$

$$\text{(D):} \qquad \underset{u,v}{\text{Minimize}} \quad \{f(u)/g(u) : v \geqslant 0,\ u \in X_0,\ v^T[h'(u)u - h(u)] \leqslant 0,$$

$$v^T h'(u) = g(u)f'(u) - f(u)g'(u)\}.$$

Here x, $u \in X_0$, an open convex subset of $\mathbb{R}^n$; $f : X_0 \to \mathbb{R}$, $g : X_0 \to \mathbb{R}$, $h : X_0 \to \mathbb{R}^m$ are differentiable functions, with gradients denoted $f'(x)$, $g'(x)$, and $h'(x)$. ($f'(x)$ and $g'(x)$ are row vectors; $h'(x)$ is an $m \times n$ matrix). Assume that $x \in X_0 \Rightarrow f(x) \geqslant 0$, and $g(x) \geqslant 0$, and that $g(x) = 0 \Rightarrow f(x) \neq 0$.

The functions f and g will be assumed *homogeneous* of the same degree λ; thus, for all $x \in R^n$ and all $t \in (0, \infty)$, $f(tx) = t^\lambda f(x)$, and likewise for g. From this follows Euler's relation: $f'(x)x = \lambda f(x)$. If f is concave and nonnegative, and g is convex and postive, then $f(x)/g(x)$ is pseudoconcave [2,3], and a local maximum of $f(x)/g(x)$ is a global maximum. The present proof proves the latter fact independently, and does not use pseudoconcavity.

DUALITY THEOREM: Let f and g be real differentiable functions, each homogeneous, of the same degree λ, with f concave and g convex; let h be convex and differentiable. Then $f(x)/g(x) \leqslant f(u)/g(u)$ whenever x is feasible for (P) and (u,v) is feasible for (D). Moreover, if also (P) attains a finite maximum at $x = a$, and if the Kuhn-Tucker constraint qualification holds there, then there exist $(\bar{u},\bar{v})$ feasible for (D), with $\bar{u} = a$. (Thus (D) is a dual problem to (P).)

PROOF: Let x be feasible for (P), and (u,v) for (D). Since h is convex, $h(u) + h'(u)(x-u) \leqslant h(x)$, hence, using, the constraints of (D),

$$\begin{aligned} 0 \geqslant v^T[h'(u)u - h(u)] &\geqslant v^T[h'(u)x - h(x)] \\ &\geqslant v^T h'(u)x \quad = g(u)f'(u)x - f(u)g'(u)x = \phi'(u)x, \end{aligned}$$

where $\phi(.) = g(u)f(.) - f(u)g(.)$ is concave and homogeneous, since $u \in X_0 \Rightarrow g(u) \geqslant 0$ and $f(u) \geqslant 0$. Hence $\phi'(u)x \geqslant \phi(x) - \phi(u) + \phi'(u)u = \phi(x) + (\lambda - 1)\phi(u) = \phi(x) + 0$. Thus, $0 \geqslant g(u)f(x) - f(u)g(x)$. Hence, if $g(u) > 0$ and $g(x) > 0$, then

$$f(u)/g(u) \geqslant f(x)/g(x). \tag{1}$$

The latter also follows when $g(u) = 0$, for then $f(u) \neq 0$, so $f(u) > 0$, and $f(u)/g(u) = +\infty$. But if $g(u) \neq 0$ and $g(x) = 0$, then $g(u)f(x) \leqslant 0$, but also $g(u) > 0$ and $f(x) > 0$, a contradiction.

Now let (P) attain a finite maximum at $x = a$, and let the Kuhn-Tucker constraint qualification hold there. Then the Kuhn-Tucker theorem applied to (P), or Schaible [4, Prop. 4], shows that the last constraint of (D) is satisfied for $u = a$ and some Lagrange multiplier $v = \bar{v} \geqslant 0$; and also that $\bar{v}^T h(a) = 0$. Then also

$$\begin{aligned} \bar{v}^T[h'(a)a - h(a)] &= \bar{v}^T h'(a)a = g(a)f'(a)a - f(a)ag'(a)a \\ &= g(a)\lambda f(a) - f(a)\lambda g(a) = 0, \end{aligned}$$

if we use the last constraint of (D), and homogeneity. Hence, $(a,\bar{v})$ is feasible for (D), and the objective functions of (P) and (D) are equal there.

REMARKS: If $g(x)$ is a linear function d^Tx, then ϕ is concave without the requirement that $u \in X_0 \Rightarrow f(u) \geqslant 0$. Weak duality then follows, as in [5], assuming however that $f(x)/g(x) < \infty$ whenever x is feasible for (P). If $g(x) > 0$ and $g(u) > 0$, the proof is as before. If $g(u) = 0$ and $g(x) > 0$ then $0 \geqslant -f(u)g(x)$ gives $f(u) > 0$, and $f(u)/g(u) = +\infty$; if $g(u) > 0$ and $g(x) = 0$, then $g(u)f(x) \leqslant 0$ gives $f(x) < 0$, and $f(x)/g(x) = -\infty$. If $g(u) = g(x) = 0$ then $f(x) \neq 0$ by assumption; since $f(x)/g(x) \neq +\infty$ by assumption, $f(x)/g(x) = -\infty$. So (1) follows in each case.

If f, g, and h are affine functions, then the constraints of (D) are also affine functions. Sharma and Swarup [5] consider $f(x) = c^Tx$ and $g(x) = d^Tx$ and prove a duality theorem, assuming that $d \geqslant 0$. They state an extension where, instead, $d^Tx \geqslant 0$ for each x feasible for (P); this corresponds here to $x \in X_0 \Rightarrow d^Tx \geqslant 0$. However, the dual in [5] omits the dual constraint $d^Tu \geqslant 0$. Example 2 in [5] gives the following two problems:

$$\text{(P1):} \qquad \text{Maximize} \left\{ \frac{2x_1 + 3x_2}{x_1 - x_2} : \; x_1 \geqslant 0,\; x_2 \geqslant 0,\; 2x_1 + 3x_2 \leqslant 6,\; x_1 - x_2 \geqslant 1 \right\};$$

$$(D1): \qquad \text{Minimize} \left\{ \frac{2u_1 + 3u_2}{u_1 - u_2} : u_1, u_2, v_1, v_2 \geqslant 0,\ 2v_1 - v_2 + 5u_2 \geqslant 0, \right.$$
$$\left. 3v_1 + v_2 - 5u_1 \geqslant 0,\ -6v_1 + v_2 \geqslant 0 \right\}.$$

If to $(D1)$ is adjoined the constraint $u_1 - u_2 \geqslant 0$, then $(D1)$ becomes indeed a dual to $(P1)$. However, $(P1)$ attains its optimal value of 6 at $x_1 = 9/5$, $x_2 = 4/5$; whereas $(D1)$, without the extra constraint, has a feasible solution $v_1 = 0$, $v_2 = 5$, $u_1 = 0$, $u_2 = 1$, for which the objective function has the value $-3 < 6$, contradicting duality.

Examples of nonlinear functions satisfying the hypotheses of the theorem have been given in [1]. If $h(x) = Ax - b$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, then the result of [1] is recovered.

If the constraint $h(x) \leqslant 0$ is generalized to $h(x) \in -S$, where S is a closed convex cone, then the theorem remains valid, with $v \geqslant 0$ replaced by $v \in S^*$, the dual cone of S, if the convex cone

$$[h(a) | h'(a)]^T(S^*)$$

is assumed closed. The latter hypothesis is required for the Kuhn-Tucker theorem; it holds automatically if S is polyhedral. This cone version remains valid if $\mathbb{R}^n$ is replaced by a Banach space of infinite dimension.

REFERENCES

[1] Craven, B.D., and B. Mond, "Duality for Homogeneous Fractional Programming," Cahiers du Centre d'Etude de Recherche Opérationelle *18* (4), 413-417, (1976).
[2] Mangasarian, O.L., "Nonlinear fractional programming," Journal of the Operations Research Society of Japan *12*, 1-10, (1969).
[3] Mangasarian, O.L., *Nonlinear Programming* (McGraw-Hill, New York, 1969), Chapter 9.
[4] Schaible, S., "Fractional Programming I, Duality," Management Science *22*, 858-867 (1976).
[5] Sharma, I.C., and K. Swarup, "On Duality in Linear Fractional Functionals Programming," Zeitschrift f*ür* Operations Research *16*, 91-100, (1972).

A NOTE ON A MODIFIED BLOCK REPLACEMENT POLICY FOR UNITS WITH INCREASING MARGINAL RUNNING COSTS

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ABSTRACT

The model for a modified block replacement policy (MBRP) is extended to include running costs. An illustrative example is worked out for the case when item life is exponentially distributed and marginal running cost per unit time increases linearly with the age of the item.

INTRODUCTION AND OUTLINE OF A MBRP WITH RUNNING COSTS

In a previous paper [3] we provided the theory for a modified block replacement policy (MBRP). In this note we introduce the additional assumption that it costs more to run a unit the older it becomes (for recent replacement models involving this assumption see Refs. [1], [2], [4], [6], [7], and [8]). More precisely, we will postulate a function $a(x)$, the marginal running cost per unit time of an item having age x. It is assumed that $a(x)$ is an increasing function of x. As in [3] the objective function that we wish to minimize is the expected cost per unit time, in the long run, of using a MBRP.

The objective function in this note is given by

$$C(b,t) = \frac{c_1 E_x[M_x(t)] + E_x[A_x(t)] + c_2\tilde{f}(0)}{t}, \tag{1}$$

where $A_x(t)$ is the expected running cost in a block interval of length t if the item is of age x at the beginning of the interval. All other symbols and expressions used in (1) have precisely the same meaning and description as in section 2 of [3].

The function $A_x(t)$ satisfies the modified renewal equation

$$A_x(t) = \int_0^t \bar{F}_x(u)\, a(x+u)\, du + \int_o^t A(t-u)\, dF_x(u), \tag{2}$$

where

$$A(t) = \int_0^t \bar{F}(u)\, a(u)\, [1+M(t-u)]\, du. \tag{3}$$

Equation (3) is obtained from (2) by setting $x = 0$ and then applying Theorem 3C in section 5.3 of [5]. The expected running cost for a block interval of length t is given by

$$E_x[A_x(t)] = \int_{0^+}^{b} \tilde{f}(x)\, A_x(t)\, dx + \tilde{f}(0)\, A(t). \tag{4}$$

A method for computing $\tilde{f}(x)$ and $\tilde{f}(0)$, the stationary age distribution of a unit at the beginning of a block interval, was developed in [3] under a factorization assumption for $f(t+y)$ (see formulas 12, 24, and 25 of [3] for details).

It should be noted that a MBRP with running costs reduces to a BRP with running costs when $b=0$. In this case (1) becomes

$$C(0,t) = \frac{c_1 M(t) + A(t) + c_2}{t}. \tag{1'}$$

Optimal BRP policies in the presence of running costs are treated in [8] for several special cases.

AN EXAMPLE MBRP WITH RUNNING COSTS

In [3] we illustrated the computation of $\tilde{f}(x)$ and $\tilde{f}(0)$ when item life follows a two-stage Erlang distribution. These results can be used to obtain $E_x[A_x(t)]$ and hence $C(b,t)$ for this particular distribution. The detailed calculations and the subsequent evaluation of (b^*,t^*) are left to the reader.

In order to highlight the way in which the consideration of running costs may affect the choice of (b^*,t^*) we shall use another example involving the exponential lifetime p.d.f. $f(x) = \lambda e^{-\lambda x}$, $x \geqslant 0$. In this case it is only the assumption of an increasing marginal running cost which justifies carrying out planned replacements. The particular running cost function which we shall use in the sequel is $a(x) = \rho x$, $x \geqslant 0$, with $\rho > 0$.

For the exponential distribution, for which the factorization of $f(t+y)$ holds trivially, we readily obtain

$$\tilde{f}(x) = \lambda e^{-\lambda x},\ 0 < x \leqslant b;\ \tilde{f}(0) = e^{-\lambda b}, \tag{5}$$

a result which could have been obtained as a direct consequence of the memoryless property of the exponential distribution. Hence,

$$M_x(t) = M(t) = \lambda t,\ t \geqslant 0. \tag{6}$$

Direct substitution in (4) yields

$$A(t) = \frac{\rho}{\lambda^2}\,[-1 + \lambda t + e^{-\lambda t}]. \tag{7}$$

Inserting (7) in (2), we obtain

$$A_x(t) = A(t) + \frac{\rho x}{\lambda}\,(1 - e^{-\lambda t}). \tag{8}$$

If we substitute (7) and (8) in (4) we get

$$E_x[A_x(t)] = \frac{\rho}{\lambda^2}\,[\lambda t - e^{-\lambda b}(1+\lambda b)(1-e^{-\lambda t})]. \tag{9}$$

The substitution of (5), (6), and (9) in (1) yields

$$C(b,t) = \lambda c_1 + \frac{\rho}{\lambda} + \frac{1}{t}\left[-\frac{\rho}{\lambda^2}\, e^{-\lambda b}(1+\lambda b)(1-e^{-\lambda t}) + c_2 e^{-\lambda b}\right]. \tag{10}$$

Taking partial derivatives of $C(b,t)$ with respect to b and t and equating to zero, we obtain the pair of equations

$$b = c_2\lambda/\rho(1-e^{-\lambda t}) \tag{11}$$

and

$$\lambda^2 c_2/\rho = (1+\lambda b)\,[1-e^{-\lambda t}(1+\lambda t)]. \tag{12}$$

Substituting (11) into (12), we obtain the following equation in t:

$$\lambda^2 c_2/\rho = (e^{\lambda t}-1)\,[1-e^{-\lambda t}(1+\lambda t)]/\lambda t. \tag{13}$$

The right hand side of (13) is a monotonically increasing function of t going from zero, when $t = 0$, to infinity, as $t \to \infty$. Hence (13) has a unique solution t^*. Inserting t^* into (11) we get b^*, which is also unique. An analysis of (11) and (13) shows that b^* and t^* are each increasing functions of c_2/ρ. In particular, $b^* \to \infty$ and $t^* \to \infty$ when $c_2/\rho \to \infty$, and $b^* \to 0$ and $t^* \to 0$ when $c_2/\rho \to 0$.

Recalling that a BRP is a special case of a MBRP when $b = 0$, we set $b = 0$ in (12) and obtain

$$\lambda^2 c_2/\rho = 1 - e^{-\lambda t}(1+\lambda t) \tag{14}$$

as the equation for t_0, the optimal block interval for the BRP.

The right hand side of (14) is a monotonically increasing function of t going from zero, for $t = 0$, to one, for $t \to \infty$. Hence a unique finite solution t_0 exists if $\lambda^2 c_2/\rho < 1$. Otherwise, $t_0 = \infty$ and the optimal BRP becomes a failure replacement policy (FRP). It is worthwhile to note that unlike the BRP, there is always a finite solution for a MBRP. It is evident that we can always do better with an optimal MBRP than with an optimal BRP. We have examined numerically the possible savings for the case of an exponential life distribution with $c_1 = c_2 = c = 0.1\ (0.1)\ 0.9$ and $\rho = 1$. It turns out that the greatest savings in using the best MBRP rather than the best BRP is attained at $c = 0.7$ and that this savings is almost 8%. We would, of course, expect even higher savings with increasing failure rate (IFR) lifetime distributions and with marginal running cost functions which increase faster than the linear function considered here.

REFERENCES

[1] Berg, M., "Optimal Replacement Policies for Two Unit Machines with Increasing Running Costs — I," Stochastic Processes and Their Applications *4*, 89-106 (1976).

[2] Berg, M., "Optimal Replacement Policies for Two Unit Machines with Increasing Running Costs — II," Stochastic Processes and Their Applications *5*, 315-322 (1977).

[3] Berg, M., and Epstein, B., "A Modified Block Replacement Policy," Naval Research Logistics Quarterly *23*, 15-24 (1976).

[4] Cléroux, R., and Hanscom, M., "Age Replacement with Adjustment and Depreciation Costs and Interest Charges," Technometrics, *16*, 235-239 (1974).

[5] Parzen, E., *Stochastic Processes* (Holden-Day, San Francisco, 1962).

[6] Ran, A., and Rosenlund, S.I., "Age Replacement with Discounting for a Continuous Maintenance Model," Technometrics *18*, 459-465 (1976).
[7] Schaeffer, R.L., "Optimum Age Replacement Policies with an Increasing Cost Factor," Technometrics *13*, 139-144 (1971).
[8] Tilquin, C., and Cléroux, R., "Block Replacement Policies with General Cost Structures," Technometrics *17*, 291-298 (1975).

A NOTE ON OPTIMAL INVENTORY MANAGEMENT UNDER INFLATION

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ABSTRACT

This paper develops a discounted-cost model that is similar to the classical economic order quantity model but includes inflation rates as parameters of the inventory system. A numerical problem is solved to illustrate the effects.

Most of the literature in the field of inventory management has not included inflation as a parameter of the system. This has happened mostly because of the belief that inflation (which was quite low in the United States prior to the 1970's) would not influence the policy variables to any significant degree. In 1975, Misra [5] and Buzacott [2] developed economic-order-quantity (EOQ) models which incorporated inflationary effects into the model. The models assume a uniform inflation rate for all the costs and minimize the average annual cost to derive an expression for the EOQ. It was also shown that if the unit selling price is changed only at the beginning of each cycle (as practiced by many grocery stores, i.e., charge more if you pay more), the objective function should be maximization of profit instead of minimization of cost. In the situation in which the selling price is increased continuously at the inflation rate, minimizing cost also maximizes profit. In a recent paper, Bierman and Thomas [1] have proposed an inflation model for the EOQ which also considers the time value of money. They too have assumed a single inflation rate for all cost factors. The cost equation in the model does not lend itself to the derivation of an expression for the EOQ, therefore the authors have suggested the use of a search method. Misra and Wortham [6] encountered a similar cost equation and suggested an approximation to derive an expression for the EOQ. For various problems the approximate EOQ was found to be within 1% of the exact EOQ. The purpose of this paper is to present a model which considers the time value of money and different inflation rates for various costs associated with an inventory system.

THE PROPOSED GENERAL EOQ MODEL

In the analysis of an inventory system, normally three types of costs are considered. These are replenishment cost, inventory carrying cost, and shortage cost. In the basic model shortages are not allowed, so only the first two costs are included in the analysis. Purchasing cost is not included, because it is constant. This is not so if we consider inflation, hence this cost will be included in the analysis. The most general and realistic model will be the one which considers a separate inflation rate for each of its cost components [3,4]. Writing a cost expression for such a model is straightforward, but its optimization is very difficult, and will require the use of search procedures [7]. However, one can put these costs into two categories;

category 1 consists of all those costs which increase at the inflation rate that prevails in the company, and category 2 consists of those that increase at the inflation rate of the general economy or of the supplier company. These will be called the internal (company) and external inflation rates respectively. Their values can be arrived at by some form of averaging (simple or weighted) of the individual inflation rates of costs in each category.

In general, replenishment cost will increase at the internal inflation rate and the unit purchasing cost at the external inflation rate. The cost of carrying inventory consists of the opportunity cost and the real out-of-pocket costs such as costs of insurance, taxes, and costs of storage. The amount of capital tied up in inventory changes with the unit cost, which increases with the external inflation rate. The cost of storage can be in either category or in both, depending on whether the company owns the storage space, or rents it, or both. Van Hees and Monhemius [4, pp. 81-101] have given an excellent breakdown of the various costs, which can be used as a guide in categorizing them along the lines suggested here. The classification would also vary depending on whether the goods are ordered from outside or are manufactured within the company. For instance, if goods are manufactured within the company, the unit cost is governed by both the internal and external inflation rates. This is so because part of the unit cost (material cost, for instance) increases with the external rate and part (setup cost + direct costs incurred in production) with the internal rate. Thus, while a clear-cut categorization of these costs is generally not possible, for a given inventory system it can easily be done. In the formulation that follows, it is assumed that this has been done and the corresponding costs determined. Also, it will be assumed that the costs vary with instantaneous inventory level. One can include additional terms if some costs depend on the maximum inventory.

Formulation

The present worth of the total cost for the first cycle is

$$P_1 = Qc + A + c_1\int_0^{Q/\lambda} (Q-\lambda t)e^{i_1 t}e^{-rt}\,dt + c_2\int_0^{Q/\lambda} (Q-\lambda t)e^{i_2 t}e^{-rt}\,dt, \tag{1}$$

where Q = reorder quantity, λ = demand per unit time, A = ordering cost, c = unit cost, c_1 = internal inventory cost per unit per unit time, i_1 = internal inflation rate, r = discount rate or cost of capital, c_2 = external inventory cost per unit per unit time, and i_2 = external inflation rate.

Equation (1) simplifies to

$$P_1 = Qc + A + \frac{c_1 Q}{R_1} - \frac{c_1\lambda}{R_1^2}(1-e^{-R_1 Q/\lambda}) + \frac{Qc_2}{R_2} - \frac{\lambda c_2}{R_2^2}(1-e^{-R_2 Q/\lambda}), \tag{2}$$

where $R_1 = r - i_1$ and $R_2 = r - i_2$.

For convenience let us define E_1 and E_2 such that

$$E_1 = A + \frac{c_1 Q}{R_1} - \frac{c_1\lambda}{R_1^2}(1-e^{-R_1 Q/\lambda}), \quad E_2 = Qc + \frac{Qc_2}{R_2} - \frac{\lambda c_2}{R_2^2}(1-e^{-R_2 Q/\lambda}),$$

The cost diagram for N cycles is

1	2		N
E_1	$E_1e^{i_1t}$	$E_1e^{2i_1t}$	$E_1e^{(N-1)i_1t}$
+	+	+	+
E_2	$E_2e^{i_2t}$	$E_2e^{2i_2t}$	$E_2e^{(N-1)i_2t}$

The present worth of the total cost for N cycles is

$$(3) \qquad P_T = E_1\left(1+e^{-R_1t}+e^{-2R_1t}+\dots\right)+E_2\left(1+e^{-R_2t}+e^{-2R_2t}+\dots\right)$$

$$= \left(\frac{A+c_1Q/R_1}{1-e^{-R_1Q/\lambda}}-\frac{c_1\lambda}{R_1^2}\right)\left(1-e^{-R_1\frac{Q}{\lambda}N}\right)+\left(\frac{Qc+Qc_2/R_2}{1-e^{-R_2Q/\lambda}}-\frac{c_2\lambda}{R_2^2}\right)\left(1-e^{-R_2\frac{Q}{\lambda}N}\right)$$

The total cost in equation (3) will converge if R_1 and R_2 are positive, i.e., the inflation rates are smaller than the discount rate, even for the infinite planning horizon, $N \to \infty$.

If the inflation rates are higher than the discount rate, the total-cost equation (3) is unbounded as $N \to \infty$. Thus, a finite horizon must be used for optimization, which we accomplish by differentiating equation (6) with respect to Q, equating it to zero, and solving for Q. This yields a complicated expression which cannot be solved for Q directly and requires the use of search techniques. In this situation, if the costs c_1 and c_2 are zero, it is optimal to have Q as large as possible. This is not a stable situation. To have finite Q, either the inflation rates should be less than the discount rate, or c_1 and c_2 should be very high. The length of planning horizon N will be determined by the forecast of the period before which the inflation rates will become less than the discount rate. In a planning horizon of unit time there are λ/Q cycles, i.e., $N = \lambda/Q$. For this case equation (3) yields

$$(4) \qquad P_T = \left(\frac{A+c_1Q/R_1}{1-e^{-R_1Q/\lambda}}-\frac{c_1\lambda}{R_1^2}\right)(1-e^{-R_1})+\left(\frac{Qc'}{1-e^{-R_2Q/\lambda}}-\frac{c_2\lambda}{R_2^2}\right)(1-e^{-R_2}),$$

where $c' = c + \dfrac{c_2}{R_2}$.

Case when (i_1 and i_2) $< r$

In this case, the cost equation (4) will be differentiated with respect to Q and equated to zero. This yields

$$(5) \qquad K\cdot\frac{\frac{c_1}{R_1}(1-e^{-R_1Q/\lambda})-(A+c_1Q/R_1)\frac{R_1}{\lambda}e^{-R_1Q/\lambda}}{(1-e^{-R_1Q/\lambda})^2}$$

$$+\frac{c'(1-e^{-R_2Q/\lambda})-Qc'\frac{R_2}{\lambda}\cdot e^{-R_2Q/\lambda}}{(1-e^{-R_2Q/\lambda})^2}=0,$$

where $K = (1-e^{-R_1})/(1-e^{-R_2})$.

Equation (5) can be solved by the use of search techniques. However, an approximate analytical solution can be obtained if we expand the exponential terms up to the first three terms and neglect the higher-order terms. This approximation has been found to yield good results in other situations with similar expressions [6]. After considerable simplification equation (5) reduces to

$$\frac{c'}{2\lambda} - \frac{AK}{R_1 Q^2} + \frac{c_1 K}{2R_1\lambda} \approx 0 . \tag{6}$$

Equation (6) yields

$$Q^* = \sqrt{\frac{2\lambda A}{I'c}}, \text{ where } I' = R_1\left(1+\frac{c_2}{cR_2} + \frac{c_1 K}{cR_1}\right)/K . \tag{7}$$

I' can be called an adjusted inventory carrying cost, following the terminology of Hadley and Whitin [2]. Thus, in practice all that is needed is to calculate I' and use it in place of I in the Harris-Wilson-Camp formula [2,3]. This does not give the optimum Q, but the approximation is quite good as will be seen later in an example. To find the optimum Q by search techniques, we can use this approximate value as a starting point.

EXAMPLE

Given $\lambda = 10{,}000$ units/year, $A = \$40$, $c = \$4.00$, $r = 0.20$, $i_1 = 0.08$, $i_2 = 0.14$, $c_1 = \$0.20$ per unit time, and $c_2 = \$0.16$ per unit per unit time, then $I' = 0.153$, and

$$Q^* \approx \sqrt{\frac{2\times 10{,}000\times 40}{0.153\times 4}} = 1148 .$$

To check the accuracy of the approximation, the exact value of Q was calculated from equation (5) by trial and error. The exact value of Q obtained was 1160, thus the approximation is quite good.

The corresponding Q from the Harris-Wilson-Camp formula is

$$Q^*_w = \sqrt{\frac{2\times 10{,}000\times 40}{4\times 0.2+0.20+0.16}} = 831 .$$

Thus, as a result of inflation the optimum order quantity has increased. The corresponding costs are

$$P_T = \$33{,}106 \text{ for } Q = 831$$

and

$$P_T = \$32{,}822 \text{ for } Q = 1148 .$$

In summary, the optimum order quantity is changed significantly when inflation is included in the analysis. However, the reduction in costs is slight. The cost function in the EOQ model is known to be insensitive in the neighborhood of the optimum Q. It is even less sensitive when given in present-worth terms. A further extension of this research is the interesting case in which lead time is significant. Since the time value of money is considered in the model, the payment policy, i.e., whether the payments are made in advance or at the time of delivery, will influence the model.

REFERENCES

[1] Bierman, Harold, and J. Thomas, "Inventory Decisions Under Inflationary Conditions," Decision Sciences *8*, No. 1 (1977).

[2] Buzacott, J.A., "Economic Order Quantities with Inflation." Operational Research Quarterly *26*, No. 3 (1975).

[3] Hadley, G., and T.M. Whitin, *Analysis of Inventory Systems*, (Prentice-Hall, Englewood Cliffs, New Jersey, 1963).

[4] Van Hees, R.N. and W. Monhemius, "*Production and Inventory Control: Theory and Practice*," (Barnes and Noble, New York, 1972) pp. 81-101.

[5] Misra, R.B., "A Study of Inflationary Effects on Inventory Systems," Logistics Spectrum *9*, No. 3 (1975).

[6] Misra, R.B., and A.W. Wortham, "The EOQ Model with Continuous Compounding," OMEGA — The International Journal of Management Science *5*, No. 1 (1977).

[7] Wilde, D.J., "*Optimum Seeking Methods*" (Prentice-Hall, Englewood Cliffs, New Jersey, 1964).

NEWS AND MEMORANDA

THE 1978 LANCHESTER PRIZE

Call for Nominations

Each year since 1954 the Council of the Operations Research Society of America has offered the Lanchester Prize for the best English-language published contribution in operations research. The Prize for 1978 consists of $2,000 and a commemorative medallion.

The screening of books and papers for the 1978 Prize will be carried out by a committee appointed by the Council of the Society. To be eligible for consideration, the book or paper must be nominated to the Committee. Nominations may be made by anyone; this notice constitutes a call for nominations.

To be eligible for the Lanchester Prize, a book, a paper or a group of books or papers must meet the following requirements:

(1) It must be on an operations research subject,

(2) It must carry a current award year publication date or, if a group, at least one member of the group must carry a current award year publication date,

(3) It must be written in the English language, and

(4) It must have appeared in the open literature.

The book(s) or paper(s) may be a case history, a report of research representing new results, or primarily expository.

For any nominated set (e.g., article and/or book) covering more than the most recent year, it is expected that each element in the set represents work from one continuous effort, such as a multi-year project or a continuously-written, multi-volume book.

Judgments will be made by the Committee using the following criteria:

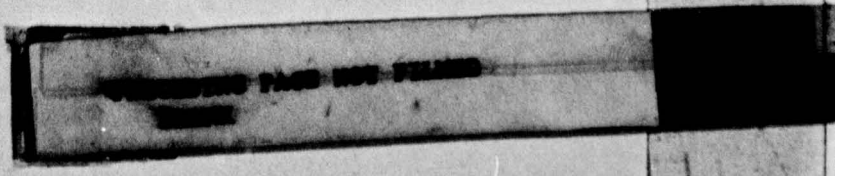
(1) The magnitude of the contribution to the advancement of the state of the art of operations research,

(2) The originality of the ideas or methods,

(3) New vistas of application opened up,

(4) The degree to which unification or simplification of existing theory or method is achieved, and

(5) Expository clarity and excellence.



Nominations should be sent to:

George L. Nemhauser, Chairman
1978 Lanchester Prize Committee
School of Operations Research
and Industrial Engineering, Upson Hall
Cornell University
Ithaca, NY 14853

Nominations may be in any form, but must include as a minimum the title(s) of paper(s) or book, author(s), place and date of publication, and six copies of the material. Supporting statements bearing on the worthwhileness of the publication in terms of the five criteria will be helpful, but are not required. Each nomination will be carefully screened by the Committee. Nominations must be received by May 30, 1979, to allow time for adequate review.

Announcement of the results of the Committee and ORSA Council action, as well as award of any prize(s) approved, will be made at the 56th National Meeting of the Society, October 21-24, 1979, in Milwaukee.

1978 Lanchester Prize Committee

Professor George Nemhauser, Chairman; *Dr. Daniel Heyman*, Dr. Ralph Keeney, Professor Leonard Kleinrock, Professor Peter Kolesar, Professor Donald Ratliff.

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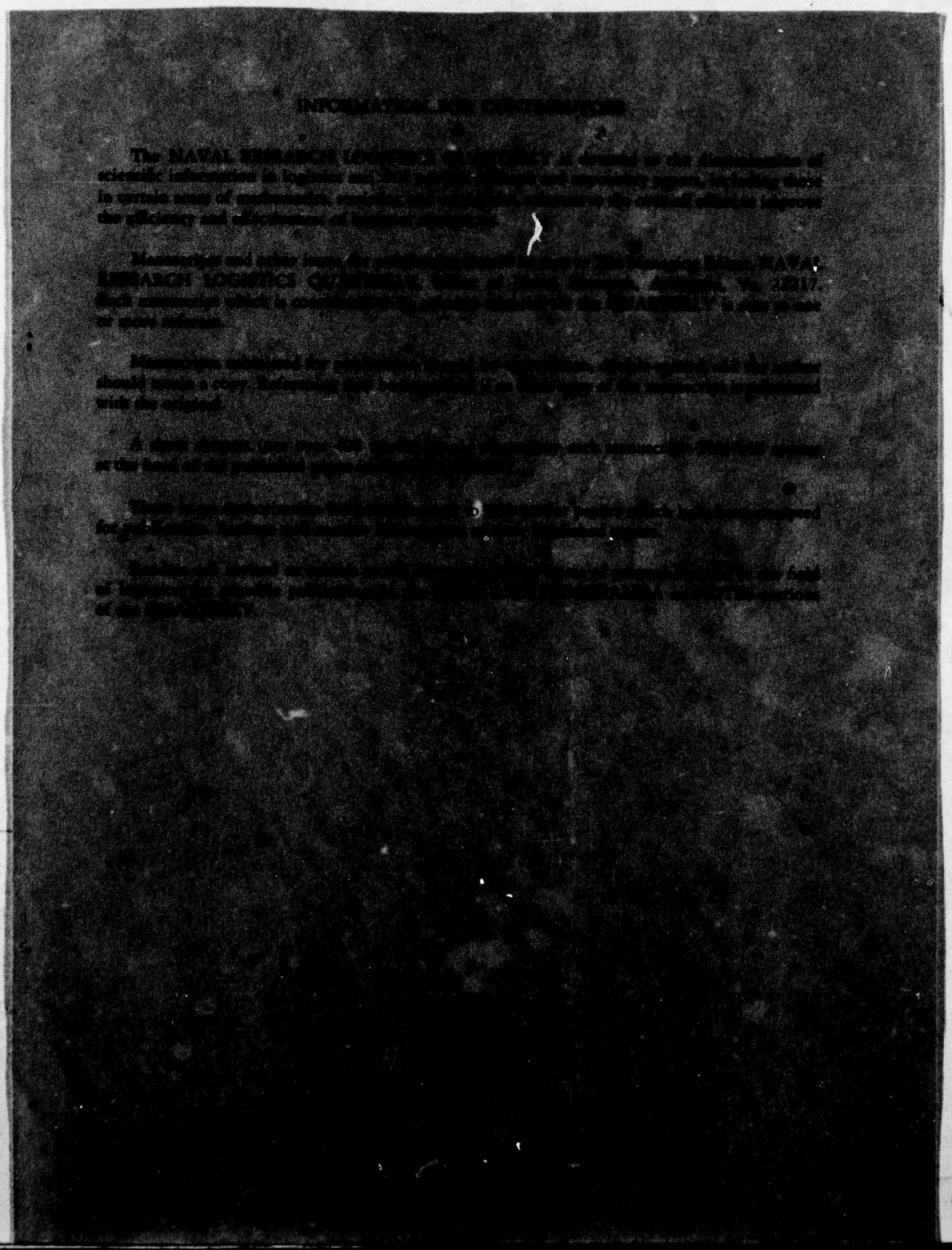

NAVAL RESEARCH
LOGISTICS
QUARTERLY

MARCH 1976
VOL. 23, NO. 1
NAVSO P-1278

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